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PROBLEMS AND THEOREMS

IN

PLANE TRIGONOMETRY.

PROBLEMS AND THEOREMS

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PLANE TRIGONOMETRY.

BY

THE REV. DAVID HEWITT, M.A.
TRINITY COLLEGE, CAMBRIDGE.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS,
FOR J. & J. J. DEIGHTON, AND T. STEVENSON.
LONDON:
JOHN W. PARKER.
M.DCCC.XL.

921.

BIBLIOTHECA · BODLEIANA ·

PREFACE.

THE Deductions in this work, which is chiefly intended to assist the young mathematician, in the solution of Plane Trigonometrical Problems, are with a very few exceptions original; though they have purposely been rendered similar to those which are proposed at College Examinations. The leading propositions on which they are founded may be met with in any of the Elementary Treatises, and are here inserted for the greater convenience of the Student. For Problems and Theorems, in those branches in which the work is deficient, the Student is referred to the copious selection (at the end of Mr Hind's Trigonometry), which he will find no difficulty in working, when he thoroughly understands the principles on which the present ones are formed.

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SYMBOLS, &c.

IN any plane triangle, unless otherwise expressed,

A, B, C are the angles,

a, b, c .... corresponding opposite sides,

S is the semi-perimeter,

K...... area,

R_1, R_2, are the radii of the inscribed and circumscribing circles.

In a right-angled triangle C is always the right angle,

θ, ϕ, ψ, χ are circular arcs,

π is 180°,

3.141592653590 is the area of a circle whose radius is 1,

ϵ is the base of Napier's system of logarithms,

S_1, S_2, are the sums of series,

S_1^{-1}, S_2^{-1} denote $\frac{1}{S_1}$ and $\frac{1}{S_2}$,

$\sin^{-1} m$, $\cos^{-1} n$, &c. denote the arcs whose sine, cosine, &c. are m, n, &c.

The radius of the circle is always supposed to be unity.

FUNDAMENTAL PROPOSITIONS.

---

(1) $\text{Sin}\,\theta = \cos\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2} + \theta\right).$

(2) $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right).$

(3) $\sin\theta = \sin(\pi - \theta) = -\sin(\pi + \theta) = \sin(2\pi + \theta) = -\sin(2\pi - \theta).$

(4) $\cos\theta = -\cos(\pi - \theta) = -\cos(\pi + \theta) = \cos(2\pi + \theta) = \cos(2\pi - \theta).$

(5) $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) = -\cot\left(\frac{\pi}{2} + \theta\right) = -\tan(\pi - \theta).$

(6) $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right) = -\tan\left(\frac{\pi}{2} + \theta\right) = -\cot(\pi - \theta).$

(7) $\sec\theta = \text{cosec}\left(\frac{\pi}{2} - \theta\right) = -\text{cosec}\left(\frac{\pi}{2} + \theta\right)$

$$= -\sec(\pi - \theta) = -\sec(\pi + \theta).$$

(8) $\text{cosec}\,\theta = \sec\left(\frac{\pi}{2} - \theta\right) = -\sec\left(\frac{\pi}{2} + \theta\right) = \text{cosec}(\pi - \theta).$

(9) $\sqrt{(4 - \text{chd}^2\theta)} = \text{chd}(\pi - \theta).$

(10) $\sin(-\theta) = -\sin\theta;\quad \cos(-\theta) = \cos\theta;\quad \text{vers}(-\theta) = \text{vers}\,\theta;$

$$\tan(-\theta) = -\tan\theta.$$

$\cot(-\theta) = -\cot\theta;\quad \sec(-\theta) = \sec\theta;\quad \text{cosec}(-\theta) = -\text{cosec}\,\theta;$

$$\text{chd}(-\theta) = \text{chd}\,\theta.$$

(11) $0 = \cos\frac{\pi}{2};\quad \cos 0 = 1.$

(12) $$1 = \sin\frac{\pi}{2} = -\sin\frac{3\pi}{2} = \tan\frac{\pi}{4} = \cot\frac{\pi}{4} = -\tan\frac{3\pi}{4} = -\cot\frac{3\pi}{4}$$
$$= -\operatorname{cosec}\frac{3\pi}{2} = \operatorname{chd}\frac{\pi}{3}.$$

(13) $$2 = \sec\frac{\pi}{3} = \operatorname{cosec}\frac{\pi}{6}.$$

(14) $$\frac{1}{2} = \sin\frac{\pi}{6} = \cos\frac{\pi}{3}.$$

(15) $$\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} = \cos\frac{\pi}{4} \text{ and } \sqrt{2-\sqrt{2}} = \operatorname{chd}\frac{\pi}{4}; \quad \sqrt{2}-1 = \tan 22^\circ.30';$$
$$\sqrt{2}+1 = \tan 67^\circ.30'.$$

(16) $$\sqrt{2} = \sec\frac{\pi}{4} = \operatorname{cosec}\frac{\pi}{4} = -\sec\frac{3\pi}{4} = \operatorname{chd}\frac{\pi}{2} = \operatorname{chd}\frac{3\pi}{2}.$$

(17) $$\frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} = \cos\frac{\pi}{6}.$$

(18) $$\frac{2}{\sqrt{3}} = \sec\frac{\pi}{6} = \operatorname{cosec}\frac{\pi}{3}.$$

(19) $$\sqrt{3} = \tan\frac{\pi}{3} = \cot\frac{\pi}{6} = -\tan\left(\frac{2\pi}{3}\right).$$

(20) $$\frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} = \cot\frac{\pi}{3}.$$

(21) $$2+\sqrt{3} = \tan 75; \quad 2-\sqrt{3} = \tan 15; \quad \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75 = \cos 15,$$
$$\text{and } \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15 = \cos 75.$$

(22) $$\frac{\sqrt{5}+1}{4} = \sin 54 = \cos 36; \quad \frac{\sqrt{5}-1}{4} = \sin 18 = \cos 72;$$
$$\frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72; \quad \frac{\sqrt{10-2\sqrt{5}}}{4} = \sin 36.$$

(23) $$1 = \sin^2\theta + \cos^2\theta = \sec^2\theta - \tan^2\theta = \operatorname{cosec}^2\theta - \cot^2\theta.$$
$$= \sin\theta \,.\, \operatorname{cosec}\theta = \cos\theta \,.\, \sec\theta = \tan\theta \,.\, \cot\theta.$$

$$\text{vers}\,\theta = 1 - \cos\theta. \tag{24}$$

$$\left\{\begin{array}{l} \text{chd}\,\theta = \sqrt{2\,.\,\text{vers}\,\theta} = \sqrt{2 - 2\,.\,\cos\theta} = 2\sin\dfrac{\theta}{2}, \\[2ex] \cos\theta = \dfrac{\cot\theta}{\text{cosec}\,\theta} = \dfrac{2 - \text{chd}^2\theta}{2}; \quad \sin\theta = \dfrac{\tan\theta}{\sec\theta}. \end{array}\right. \tag{25}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{26}$$

$$\cot\theta + \tan\theta = 2\,\text{cosec}\,2\theta; \quad \cot\theta - \tan\theta = 2\cot 2\theta. \tag{27}$$

$$\tan\theta = \text{cosec}\,2\theta - \cot 2\theta; \quad \cot\theta = \text{cosec}\,2\theta + \cot 2\theta. \tag{28}$$

$$\sin 2\theta = 2\sin\theta\,.\,\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta}. \tag{29}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}. \tag{30}$$

$$\cot^2\theta = \frac{1 + \cos 2\theta}{1 - \cos 2\theta}. \tag{31}$$

$$1 + \cos 2\theta = 2\cos^2\theta; \quad 1 - \cos 2\theta = 2\sin^2\theta. \tag{32}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}. \tag{32}$$

$$\tan 2\theta = \frac{2\,.\,\tan\theta}{1 - \tan^2\theta}. \tag{33}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta; \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta. \tag{34}$$

$$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}. \tag{35}$$

$$\sin(\theta \pm \phi) = \sin\theta\,.\,\cos\phi \pm \cos\theta\,.\,\sin\phi. \tag{36}$$

$$\cos(\theta \pm \phi) = \cos\theta\,.\,\cos\phi \mp \sin\theta\,.\,\sin\phi. \tag{37}$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\,.\,\tan\phi} \text{ and } \cot(\theta \pm \phi) = \frac{\cot\theta\,.\,\cot\phi \mp 1}{\cot\phi \pm \cot\theta}. \tag{38}$$

(39) $$\sec^{-1}\theta \pm \sec^{-1}\phi = \sec^{-1}\left(\frac{\sec\theta\,.\,\sec\phi}{1 \mp \sqrt{\sec^2\theta - 1}\,.\,\sqrt{\sec^2\phi - 1}}\right);$$

and $$\operatorname{cosec}^{-1}\theta \pm \operatorname{cosec}^{-1}\phi = \operatorname{cosec}^{-1}\left(\frac{\operatorname{cosec}\theta\,.\,\operatorname{cosec}\phi}{\sqrt{\operatorname{cosec}^2\phi - 1} \pm \sqrt{\operatorname{cosec}^2\theta - 1}}\right).$$

(40) $$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right).\cos\left(\frac{\theta-\phi}{2}\right),$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\theta-\phi}{2}\right),$$

$$\cos\phi + \cos\theta = 2\cos\left(\frac{\theta+\phi}{2}\right).\cos\left(\frac{\theta-\phi}{2}\right),$$

$$\cos\phi - \cos\theta = 2\sin\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\theta-\phi}{2}\right).$$

(41) $$\sin(\theta+\phi).\sin(\theta-\phi) = \sin^2\theta - \sin^2\phi,$$

$$\cos(\theta+\phi).\cos(\theta-\phi) = \cos^2\theta - \sin^2\phi.$$

(42) $$\sin(\theta+\phi) + \sin(\theta-\phi) = 2\sin\theta.\cos\phi,$$

$$\sin(\theta+\phi) - \sin(\theta-\phi) = 2\cos\theta.\sin\phi,$$

$$\cos(\theta-\phi) + \cos(\theta+\phi) = 2\cos\theta.\cos\phi,$$

$$\cos(\theta-\phi) - \cos(\theta+\phi) = 2\sin\theta.\sin\phi.$$

(43) $$\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)} = \frac{\tan\theta + \tan\phi}{\tan\theta - \tan\phi}.$$

(44) $$\frac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi} = \frac{\tan\left(\frac{\theta+\phi}{2}\right)}{\tan\left(\frac{\theta-\phi}{2}\right)}.$$

(45) $$\frac{\sin\theta + \sin\phi}{\cos\theta + \cos\phi} = \tan\left(\frac{\theta+\phi}{2}\right).$$

(46) In a right-angled triangle $2R_1 = c$.

(47) In any triangle, $\tan A.\tan B.\tan C = \tan A + \tan B + \tan C$,

$$\cot A.\cot B + \cot B.\cot C + \cot C.\cot A = 1.$$

(48) In oblique-angled triangles,

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \qquad \frac{\sin B}{\sin C} = \frac{b}{c}, \qquad \frac{\sin A}{\sin C} = \frac{a}{c}.$$

(49) $$K = \frac{ab}{2} . \sin C = \frac{ac}{2} . \sin B = \frac{bc}{2} . \sin A$$
$$= \sqrt{S . (S-a) . (S-b) . (S-c)}.$$

(50) $$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{a^2+c^2-b^2}{2ac}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

(51) $$\sin\frac{A}{2} = \sqrt{\left\{\frac{(S-b).(S-c)}{bc}\right\}}, \quad \sin\frac{B}{2} = \sqrt{\left\{\frac{(S-a).(S-c)}{ac}\right\}},$$
$$\sin\frac{C}{2} = \sqrt{\left\{\frac{(S-a).(S-b)}{ab}\right\}}.$$

(52) $$\cos\frac{A}{2} = \sqrt{\left\{\frac{S.(S-a)}{bc}\right\}}, \quad \cos\frac{B}{2} = \sqrt{\left\{\frac{S.(S-b)}{ac}\right\}},$$
$$\cos\frac{C}{2} = \sqrt{\left\{\frac{S.(S-c)}{ab}\right\}}.$$

(53) $$\cot\frac{A}{2} = \sqrt{\left\{\frac{S.(S-a)}{(S-b).(S-c)}\right\}}, \quad \cot\frac{B}{2} = \sqrt{\left\{\frac{S.(S-b)}{(S-a).(S-c)}\right\}},$$
$$\cot\frac{C}{2} = \sqrt{\left\{\frac{S.(S-c)}{(S-a).(S-b)}\right\}}.$$

(54) $$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}.$$

(55) N degrees $= N + \frac{N}{9}$ grades; N grades $= N - \frac{N}{10}$ degrees.

(56) $\cos n\theta \pm \sqrt{-1} . \sin n\theta = (\cos\theta \pm \sqrt{-1} . \sin\theta)^n$, where n may be either positive or negative, integral or fractional.

(57) If $2\cos\theta = x + \frac{1}{x}$, then $2\cos n\theta = x^n + \frac{1}{x^n}$, where n may be either integral or fractional, positive or negative.

(58) $$\cos\theta = \frac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2}, \qquad \sin\theta = \frac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}.$$

(59) $$\theta = \tan\theta - \frac{1}{3} . \tan^3\theta + \frac{1}{5} . \tan^5\theta + \frac{1}{7} . \tan^7\theta + \ldots\ldots$$

(60) $$\theta = \sin\theta \,.\, \sec\frac{\theta}{2} \,.\, \sec\frac{\theta}{4} \,.\, \sec\frac{\theta}{8} \ldots\ldots\ldots$$

(61) $$\frac{\theta}{2} = \sin\theta - \frac{1}{2}\,.\,\sin 2\theta + \frac{1}{3}\,.\,\sin 3\theta - \frac{1}{4}\,.\,\sin 4\theta + \ldots\ldots\ldots$$

(62) $$\sin\theta = \theta - \frac{\theta^3}{1\,.\,2\,.\,3} + \frac{\theta^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \ldots\ldots\ldots$$

(63) $$\cos\theta = 1 - \frac{\theta^2}{1\,.\,2} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \frac{\theta^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} + \ldots\ldots\ldots$$

(64) $$\tan\theta = \theta + \frac{\theta^3}{1\,.\,3} + \frac{2\theta^5}{1\,.\,3\,.\,5} + \ldots\ldots\ldots$$

(65) $$\sec\theta = 1 + \frac{\theta^2}{1\,.\,2} + \frac{5\theta^4}{1\,.\,2\,.\,3\,.\,4} + \ldots\ldots\ldots$$

(66) $$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots \cos(2n-1)\theta = \frac{\cos n\theta \,.\, \sin n\theta}{\sin\theta}.$$

(67) $$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots\ldots \cos n\theta$$

$$= \frac{\cos\left(\frac{n+1}{2}\right)\theta \,.\, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

(68) $$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots\ldots \sin n\theta$$

$$= \frac{\sin\left(\frac{n+1}{2}\right)\theta \,.\, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

(69) $$\cos\theta + \cos 4\theta + \cos 7\theta + \ldots\ldots\ldots \cos(3n-2)\theta$$

$$= \frac{\cos\left(\frac{3n-1}{2}\right)\theta \,.\, \sin\frac{3n\theta}{2}}{\sin\frac{3\theta}{2}}.$$

(70) $$\sin\theta + \sin 4\theta + \sin 7\theta + \ldots\ldots\ldots \sin(3n-2)\theta$$

$$= \frac{\sin\left(\frac{3n-1}{2}\right)\theta \,.\, \sin\frac{3n\theta}{2}}{\sin\frac{3\theta}{2}}.$$

(71) $\log x^n = n \log x, \qquad \log x^{\frac{1}{n}} = \frac{\log x}{n}.$

(72) $\log (xy) = \log x + \log y,$ and $\log \left(\frac{x}{y}\right) = \log x - \log y.$

(73) $\log_\epsilon \epsilon^x = x.$

(74) $\epsilon^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} + \ldots\ldots\ldots$

$$\epsilon^{-x} = 1 - x + \frac{x^2}{1 . 2} - \frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} - \ldots\ldots\ldots$$

(75) $\log_\epsilon (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

$$\log_\epsilon (1 - x) = - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots$$

CHAPTER I.

FORMULÆ.

1. $$\text{Tan}\,(\pi+\theta)=\tan\,(2\pi+\theta),$$

$$\tan\,(2\pi+\theta)=\frac{\sin\,(2\pi+\theta)}{\cos\,(2\pi+\theta)}\quad(26)$$

$$=\frac{\sin\theta}{\cos\theta}\quad(3)\text{ and }(4).$$

$$\text{Also }\tan\,(\pi+\theta)=\frac{-\sin\,(\pi+\theta)}{-\cos\,(\pi+\theta)}\quad(26)$$

$$=\frac{\sin\theta}{\cos\theta}\quad(3)\text{ and }(4);$$

$$\therefore\ \tan\,(\pi+\theta)=\tan\,(2\pi+\theta).$$

2. $$\sin\left(\frac{5\pi}{2}+\theta\right)-\sin\left(\frac{3\pi}{2}-\theta\right)=\sin\left(\frac{5\pi}{2}-\theta\right)-\sin\left(\frac{3\pi}{2}+\theta\right),$$

$$\sin\left(\frac{5\pi}{2}+\theta\right)=\sin\left\{2\pi+\left(\frac{\pi}{2}+\theta\right)\right\}$$

$$=\sin\left(\frac{\pi}{2}+\theta\right)\quad(3)$$

$$=\cos\theta\qquad(2);$$

$$\text{and }\sin\left(\frac{3\pi}{2}-\theta\right)=\sin\left\{\pi+\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$=-\sin\left(\frac{\pi}{2}-\theta\right)\quad(3)$$

$$=-\cos\theta\qquad(2);$$

$$\therefore\ \sin\left(\frac{5\pi}{2}+\theta\right)-\sin\left(\frac{3\pi}{2}-\theta\right)=2\cos\theta.$$

Similarly it may be shewn that

$$\sin\left(\frac{5\pi}{2}-\theta\right)-\sin\left(\frac{3\pi}{2}+\theta\right)=2\cos\theta,$$

and $\therefore \sin\left(\frac{5\pi}{2}+\theta\right)-\sin\left(\frac{3\pi}{2}-\theta\right)=\sin\left(\frac{5\pi}{2}-\theta\right)-\sin\left(\frac{3\pi}{2}+\theta\right)$.

3. $$\cot\left(\frac{\pi}{2}+\theta\right)-\tan\left(\frac{\pi}{2}+\theta\right)=2\cot 2\theta,$$

$$\cot\left(\frac{\pi}{2}+\theta\right)=-\cot\left\{\pi-\left(\frac{\pi}{2}+\theta\right)\right\} \quad (6)$$

$$=-\cot\left(\frac{\pi}{2}-\theta\right)$$

$$=-\tan\theta \quad (5);$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=-\tan\left\{\pi-\left(\frac{\pi}{2}+\theta\right)\right\} \quad (5)$$

$$=-\tan\left(\frac{\pi}{2}-\theta\right)$$

$$=-\cot\theta \quad (6);$$

$$\therefore \cot\left(\frac{\pi}{2}+\theta\right)-\tan\left(\frac{\pi}{2}+\theta\right)=\cot\theta-\tan\theta$$

$$=2\cot 2\theta \quad (27).$$

4. $$\tan^2\left(\frac{\pi}{2}+\theta\right)-\cot^2\left(\frac{\pi}{2}+\theta\right)=\operatorname{cosec}^2\theta-\sec^2\theta,$$

$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$ (6); $\therefore \tan^2\left(\frac{\pi}{2}+\theta\right)=\cot^2\theta$,

and $\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta$ (5); $\therefore \cot^2\left(\frac{\pi}{2}+\theta\right)=\tan^2\theta$,

and $\therefore \tan^2\left(\frac{\pi}{2}+\theta\right)-\cot^2\left(\frac{\pi}{2}+\theta\right)=\cot^2\theta-\tan^2\theta$

$$=(\operatorname{cosec}^2\theta-1)-(\sec^2\theta-1) \quad (23)$$

$$=\operatorname{cosec}^2\theta-\sec^2\theta.$$

5. $$\frac{\sec^2\left(\frac{\pi}{2}+\theta\right)+\operatorname{cosec}^2\left(\frac{\pi}{2}+\theta\right)}{\sec^2\left(\frac{\pi}{2}+\theta\right)-\operatorname{cosec}^2\left(\frac{\pi}{2}+\theta\right)}=\sec 2\theta,$$

$$\frac{\sec^2\left(\frac{\pi}{2}+\theta\right)+\operatorname{cosec}^2\left(\frac{\pi}{2}+\theta\right)}{\sec^2\left(\frac{\pi}{2}+\theta\right)-\operatorname{cosec}^2\left(\frac{\pi}{2}+\theta\right)}=\frac{\operatorname{cosec}^2\theta+\sec^2\theta}{\operatorname{cosec}^2\theta-\sec^2\theta}\quad (7)\text{ and }(8)$$

$$=\frac{\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}-\frac{1}{\cos^2\theta}}\quad (23)$$

$$=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$

$$=\frac{1}{\cos 2\theta}\quad (23)\text{ and }(30)$$

$$=\sec 2\theta\quad (23).$$

6. $$\operatorname{vers}\left(\frac{\pi}{2}+\theta\right).\operatorname{vers}\left(\frac{\pi}{2}-\theta\right)+\operatorname{vers}\theta.\operatorname{vers}(\pi-\theta)=1,$$

$$\operatorname{vers}\left(\frac{\pi}{2}+\theta\right)=1-\cos\left(\frac{\pi}{2}+\theta\right)=1+\sin\theta,$$

$$\operatorname{vers}\left(\frac{\pi}{2}-\theta\right)=1-\cos\left(\frac{\pi}{2}-\theta\right)=1-\sin\theta\quad (24)\text{ and }(1);$$

$$\therefore\ \operatorname{vers}\left(\frac{\pi}{2}+\theta\right).\operatorname{vers}\left(\frac{\pi}{2}-\theta\right)=1-\sin^2\theta=\cos^2\theta\quad (23).$$

Again $\operatorname{vers}\theta=1-\cos\theta,$

$$\operatorname{vers}(\pi-\theta)=1-\cos(\pi-\theta)=1+\cos\theta\quad (4);$$

$$\therefore\ \operatorname{vers}\theta.\operatorname{vers}(\pi-\theta)=1-\cos^2\theta=\sin^2\theta\quad (23).$$

And $$\operatorname{vers}\left(\frac{\pi}{2}+\theta\right).\operatorname{vers}\left(\frac{\pi}{2}-\theta\right)+\operatorname{vers}\theta.\operatorname{vers}(\pi-\theta)$$

$$=\cos^2\theta+\sin^2\theta=1\quad (23).$$

7. $$\frac{\text{chd}\left(\frac{\pi}{2}+\theta\right).\,\text{chd}\left(\frac{\pi}{2}-\theta\right).\,\text{chd}\,\theta\,.\,\text{chd}\,(\pi-\theta)}{\text{vers}\left(\frac{\pi}{2}+\theta\right).\,\text{vers}\left(\frac{\pi}{2}-\theta\right)-\text{vers}\,\theta\,.\,\text{vers}\,(\pi-\theta)}=2\tan 2\theta,$$

$$\text{chd}^2\left(\frac{\pi}{2}+\theta\right).\,\text{chd}^2\left(\frac{\pi}{2}-\theta\right).\,\text{chd}^2\theta\,.\,\text{chd}^2(\pi-\theta)$$

$$=2\,\text{vers}\left(\frac{\pi}{2}+\theta\right).\,2\,\text{vers}\left(\frac{\pi}{2}-\theta\right).\,2\,\text{vers}\,\theta\,.\,2\,\text{vers}\,(\pi-\theta)\quad(25)$$

$$=16\cos^2\theta\,.\,\sin^2\theta.\quad\text{(Prop. 6)};$$

$$\therefore\ \text{chd}\left(\frac{\pi}{2}+\theta\right).\,\text{chd}\left(\frac{\pi}{2}-\theta\right).\,\text{chd}\,\theta\,.\,\text{chd}\,(\pi-\theta)$$

$$=4\cos\theta\,.\,\sin\theta=2\sin 2\theta.\quad(29).$$

Again $\text{vers}\left(\frac{\pi}{2}+\theta\right).\,\text{vers}\left(\frac{\pi}{2}-\theta\right)-\text{vers}\,\theta\,.\,\text{vers}\,(\pi-\theta)$

$$=\cos^2\theta-\sin^2\theta=\cos 2\theta.\quad\text{(Prop. 6) and (30)};$$

$$\therefore\ \frac{\text{chd}\left(\frac{\pi}{2}+\theta\right).\,\text{chd}\left(\frac{\pi}{2}-\theta\right).\,\text{chd}\,\theta\,.\,\text{chd}\,(\pi-\theta)}{\text{vers}\left(\frac{\pi}{2}+\theta\right).\,\text{vers}\left(\frac{\pi}{2}-\theta\right)-\text{vers}\,\theta\,.\,\text{vers}\,(\pi-\theta)}$$

$$=2\,.\,\frac{\sin 2\theta}{\cos 2\theta}=2\tan 2\theta\quad(26).$$

8. $$\frac{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)-\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)+\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}=\sin\theta,$$

$$\text{chd}^2\left(\frac{\pi}{2}+\theta\right)=2\,\text{vers}\left(\frac{\pi}{2}+\theta\right)\ (25)=2\left\{1-\cos\left(\frac{\pi}{2}+\theta\right)\right\}\ (24)$$

$$=2+2\sin\theta\quad(1).$$

Similarly $\text{chd}^2\left(\frac{\pi}{2}-\theta\right)=2-2\sin\theta;$

$$\therefore\ \frac{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)-\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)+\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}=\frac{4\sin\theta}{4}=\sin\theta.$$

9. $$\frac{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)-\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}{\text{chd}\left(\frac{\pi}{2}+\theta\right).\text{chd}\left(\frac{\pi}{2}-\theta\right)}=2\tan\theta.$$

$$\text{chd}^2\left(\frac{\pi}{2}+\theta\right).\text{chd}^2\left(\frac{\pi}{2}-\theta\right)=(2+2\sin\theta)(2-2\sin\theta)\quad\text{Prop. 8}$$

$$=4-4\sin^2\theta=4(1-\sin^2\theta)=4\cos^2\theta\quad(23);$$

$$\therefore\ \text{chd}\left(\frac{\pi}{2}+\theta\right).\text{chd}\left(\frac{\pi}{2}-\theta\right)=2\cos\theta,$$

and

$$\frac{\text{chd}^2\left(\frac{\pi}{2}+\theta\right)-\text{chd}^2\left(\frac{\pi}{2}-\theta\right)}{\text{chd}\left(\frac{\pi}{2}+\theta\right).\text{chd}\left(\frac{\pi}{2}-\theta\right)}=\frac{4\sin\theta}{2\cos\theta}\ (\text{Prop. 8})=2.\frac{\sin\theta}{\cos\theta}=2\tan\theta\quad(26).$$

10. $$\text{chd}(\pi-\theta)=\frac{\text{chd}\,2\theta}{\text{chd}\,\theta},$$

$$\text{chd}(\pi-\theta)=2\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\quad(25)$$

$$=2\cos\frac{\theta}{2}\quad(2)$$

$$=\frac{2.\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$=\frac{2\sin\theta}{2\sin\frac{\theta}{2}}\quad(29)$$

$$=\frac{\text{chd}\,2\theta}{\text{chd}\,\theta}\quad(25).$$

11. $$\text{vers}(\pi-\theta)=\sin\theta.\cot\frac{\theta}{2},$$

$$\text{vers}(\pi-\theta)=1-\cos(\pi-\theta)\quad(24)$$

$$=1+\cos\theta\quad(4)$$

$$=2\cos^2\frac{\theta}{2}\quad(32)$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}\right)$$

$$= \cot\frac{\theta}{2}\,.\,\sin\theta \quad (26) \text{ and } (29).$$

12. $$\frac{\text{vers}\,(\pi+\theta)}{\cos\,(2\,\pi-\theta)} = \frac{\tan\theta}{\tan\frac{\theta}{2}},$$

$$\frac{\text{vers}\,(\pi+\theta)}{\cos\,(2\,\pi-\theta)} = \frac{1+\cos\theta}{\cos\theta} \quad (24) \text{ and } (4)$$

$$= 1 + \frac{1}{\cos\theta}$$

$$= 1 + \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \quad (30)$$

$$= \frac{2}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{1}{\tan\frac{\theta}{2}}\,.\,\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{\tan\theta}{\tan\frac{\theta}{2}} \quad (33).$$

13. $$\text{cosec}\,(\pi-2\,\theta)\,.\,\text{chd}\,(\pi-2\,\theta) = \text{cosec}\,\theta,$$

$$\text{cosec}\,(\pi-2\theta) = \frac{1}{\sin\,(\pi-2\theta)} \quad (23)$$

$$= \frac{1}{\sin 2\,\theta} \quad (3)$$

$$= \frac{1}{2\sin\theta\,.\,\cos\theta} \quad (29).$$

$$\text{Also } \operatorname{chd}(\pi - 2\theta) = 2\sin\left(\frac{\pi}{2} - \theta\right) \quad (25)$$

$$= 2\cos\theta \quad (2);$$

$$\therefore \operatorname{cosec}(\pi - 2\theta)\,.\,\operatorname{chd}(\pi - 2\theta) = \frac{1}{\sin\theta} = \operatorname{cosec}\theta \quad (23).$$

14. $$\frac{2\cot\left(\frac{\pi}{4} + \theta\right).\cot\left(\frac{\pi}{4} - \theta\right)}{\operatorname{cosec}\left(\frac{\pi}{4} + \theta\right).\sec\left(\frac{\pi}{4} - \theta\right)} = 1 + \sin\theta,$$

$$\cot\left(\frac{\pi}{4} + \theta\right).\cot\left(\frac{\pi}{4} - \theta\right) = \frac{\tan\left\{\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta\right)\right\}}{\tan\left(\frac{\pi}{4} + \theta\right)} \quad (6) \text{ and } (23)$$

$$= \frac{\tan\left(\frac{\pi}{4} + \theta\right)}{\tan\left(\frac{\pi}{4} + \theta\right)}$$

$$= 1.$$

$$\text{Also } \frac{2}{\operatorname{cosec}\left(\frac{\pi}{4} + \theta\right).\sec\left(\frac{\pi}{4} - \theta\right)} = 2\sin\left(\frac{\pi}{4} + \theta\right).\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= \sin\left\{\left(\frac{\pi}{4} + \theta\right) + \left(\frac{\pi}{4} - \theta\right)\right\} + \sin\left\{\left(\frac{\pi}{4} + \theta\right) - \left(\frac{\pi}{4} - \theta\right)\right\} \quad (42)$$

$$= \sin\frac{\pi}{2} + \sin 2\theta$$

$$= 1 + \sin 2\theta \quad (12);$$

$$\therefore \frac{2\cot\left(\frac{\pi}{4} + \theta\right).\cot\left(\frac{\pi}{4} - \theta\right)}{\operatorname{cosec}\left(\frac{\pi}{4} + \theta\right).\sec\left(\frac{\pi}{4} - \theta\right)} = 1 + \sin 2\theta.$$

15. $$\frac{\operatorname{cosec}\left(\frac{\pi}{4} + \frac{\theta}{2}\right).\operatorname{cosec}\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2}} = \tan\theta,$$

$$\frac{\cot\frac{\theta}{2}+\tan\frac{\theta}{2}}{\operatorname{cosec}\left(\frac{\pi}{4}+\frac{\theta}{2}\right).\operatorname{cosec}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$=2\operatorname{cosec}\theta\,.\,\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right).\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\quad(23)\text{ and }(27)$$

$$=\frac{2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right).\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\sin\theta}\quad(1)\text{ and }(23)$$

$$=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\theta}\quad(29)$$

$$=\frac{\cos\theta}{\sin\theta};$$

$$\therefore\ \frac{\operatorname{cosec}\left(\frac{\pi}{4}+\frac{\theta}{2}\right).\operatorname{cosec}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cot\frac{\theta}{2}+\tan\frac{\theta}{2}}=\frac{\sin\theta}{\cos\theta}=\tan\theta\quad(26).$$

16. $$\frac{4\sin\left(\frac{\pi}{6}+\frac{\theta}{4}\right).\sin\left(\frac{\pi}{6}-\frac{\theta}{4}\right)}{\sin\theta}=\operatorname{cosec}\frac{\theta}{2}-\operatorname{cosec}\theta,$$

$$\sin\left(\frac{\pi}{6}+\frac{\theta}{4}\right).\sin\left(\frac{\pi}{6}-\frac{\theta}{4}\right)=\sin^2\frac{\pi}{6}-\sin^2\frac{\theta}{4}\quad(41)$$

$$=\frac{1}{4}-\sin^2\frac{\theta}{4}\quad(14);$$

$$\therefore\ 4\sin\left(\frac{\pi}{6}+\frac{\theta}{4}\right).\sin\left(\frac{\pi}{6}-\frac{\theta}{4}\right)=1-4\sin^2\frac{\theta}{4}$$

$$=1-2\left(1-\cos\frac{\theta}{2}\right)\quad(32)$$

$$=2\cos\frac{\theta}{2}-1;$$

$$\therefore\ \frac{4\sin\left(\frac{\pi}{6}+\frac{\theta}{4}\right).\sin\left(\frac{\pi}{6}-\frac{\theta}{4}\right)}{\sin\theta}=\frac{2\cos\frac{\theta}{2}}{\sin\theta}-\frac{1}{\sin\theta}$$

$$= \frac{2 \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} - \frac{1}{\sin \theta} \qquad (29)$$

$$= \frac{1}{\sin \frac{\theta}{2}} - \frac{1}{\sin \theta}$$

$$= \operatorname{cosec} \frac{\theta}{2} - \operatorname{cosec} \theta.$$

17. $(\sec \frac{\pi}{4} + \sin 2\theta) . (\sec \frac{\pi}{4} - \sin 2\theta) = 2 . (\sin^4\theta + \cos^4\theta),$

$$(\sec \frac{\pi}{4} + \sin 2\theta) . (\sec \frac{\pi}{4} - \sin 2\theta)$$

$$= (\sqrt{2} + \sin 2\theta) . (\sqrt{2} - \sin 2\theta) \qquad (16)$$

$$= 2 - \sin^2 2\theta$$

$$= 2 - 4 \sin^2\theta . \cos^2\theta \qquad (29)$$

$$= 2 (1 - 2 \sin^2\theta . \cos^2\theta)$$

$$= 2 \{ (\sin^2\theta + \cos^2\theta)^2 - 2 \sin^2\theta . \cos^2\theta \}$$

$$= 2 (\sin^4\theta + \cos^4\theta).$$

18. $\dfrac{\sec \left(\frac{\pi}{4} + n\theta\right)}{\sec \frac{\pi}{4} . \sec 2n\theta} = \sin n\theta + \cos n\theta,$

$$\frac{\sec \left(\frac{\pi}{4} + n\theta\right)}{\sec \frac{\pi}{4} . \sec 2n\theta} = \frac{\cos \frac{\pi}{4} . \cos 2n\theta}{\cos \left(\frac{\pi}{4} + n\theta\right)} \qquad (23)$$

$$= \frac{\cos \frac{\pi}{4} (\cos^2 n\theta - \sin^2 n\theta)}{\cos \frac{\pi}{4} . \cos n\theta - \sin \frac{\pi}{4} . \sin n\theta} \qquad (30) \text{ and } (37)$$

$$= \frac{(\cos n\theta + \sin n\theta) . (\cos n\theta - \sin n\theta)}{\cos n\theta - \sin n\theta} \qquad (15)$$

$$= \sin n\theta + \cos n\theta.$$

19. $(2\cot 4n\theta + \tan 2n\theta)\,.\,(2\operatorname{cosec} 2n\theta - \cot n\theta)$

$$= \tfrac{1}{2}\left(\tan\frac{\pi}{4} + \tan n\theta\right).\left(\tan\frac{\pi}{4} - \tan n\theta\right),$$

$$2\cot 4n\theta + \tan 2n\theta = \cot 2n\theta - \tan 2n\theta + \tan 2n\theta \quad (27)$$

$$= \cot 2n\theta,$$

$$2\operatorname{cosec} 2n\theta - \cot n\theta = \cot n\theta + \tan n\theta - \cot n\theta \quad (27)$$

$$= \tan n\theta;$$

$$\therefore\ (2\cot 4n\theta + \tan 2n\theta)(2\operatorname{cosec} 2n\theta - \cot n\theta)$$

$$= \cot 2n\theta\,.\,\tan n\theta$$

$$= \frac{1 - \tan^2 n\theta}{2} \qquad (23) \text{ and } (33)$$

$$= \tfrac{1}{2}(1 - \tan^2 n\theta)$$

$$= \tfrac{1}{2}(1 + \tan n\theta)\,.\,(1 - \tan n\theta)$$

$$= \tfrac{1}{2}\left(\tan\frac{\pi}{4} + \tan n\theta\right).\left(\tan\frac{\pi}{4} - \tan n\theta\right).$$

20. $\dfrac{1 + \sin 2\theta + \cos\theta + \sin\theta}{\cos 2\theta + \cos\theta + \sin\theta} = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right),$

$$\frac{1 + \sin 2\theta + \cos\theta + \sin\theta}{\cos 2\theta + \cos\theta + \sin\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\,.\,\cos\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta + \cos\theta + \sin\theta} \quad (23), (29) \text{ and } (30)$$

$$= \frac{(\cos\theta + \sin\theta)^2 + (\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)\,.\,(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}$$

$$= \frac{\cos\theta + \sin\theta + 1}{\cos\theta - \sin\theta + 1}$$

$$= \frac{2\cos^2\dfrac{\theta}{2} + 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2} - 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}} \qquad (29) \text{ and } (32)$$

$$= \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$= \frac{1 + \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1 - \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} \qquad (26)$$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4} \cdot \tan\frac{\theta}{2}} \qquad (12)$$

$$= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \qquad (38).$$

21. $\cos 2\theta + \sin 2\phi = 2\cos\left\{(\theta - \phi) + \frac{\pi}{4}\right\} \cdot \cos\left\{(\theta + \phi) - \frac{\pi}{4}\right\}$,

$$\cos 2\theta + \sin 2\phi = \cos 2\theta + \cos\left(\frac{\pi}{2} - 2\phi\right)$$

$$= 2\cos\left\{\frac{2\theta + \left(\frac{\pi}{2} - 2\phi\right)}{2}\right\} \cdot \cos\left\{\frac{2\theta - \left(\frac{\pi}{2} - 2\phi\right)}{2}\right\} \qquad (40)$$

$$= 2\cos\left\{(\theta - \phi) + \frac{\pi}{4}\right\} \cdot \cos\left\{(\theta + \phi) - \frac{\pi}{4}\right\}.$$

22. $\dfrac{\cos\frac{\theta}{2}}{\operatorname{vers}\frac{\theta}{2}} = \dfrac{\sin\theta}{\operatorname{chd}\theta - \sin\theta}$,

$$\frac{\cos\frac{\theta}{2}}{\text{vers}\,\frac{\theta}{2}}=\frac{\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}$$

$$=\frac{2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}-2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}}$$

$$=\frac{\sin\theta}{\text{chd}\,\theta-\sin\theta}\quad(25)\text{ and }(29).$$

23. $$\frac{\text{chd}^2\theta}{\text{chd}\,2\theta}=\tan\frac{\theta}{2},$$

$$\frac{\text{chd}^2\theta}{\text{chd}\,2\theta}=\frac{4\sin^2\frac{\theta}{2}}{2\sin\theta}\quad(25)$$

$$=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}}$$

$$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$=\tan\frac{\theta}{2}\quad(26).$$

24. $$\frac{\sec\theta}{\text{cosec}\,\theta\,.\,\cot\theta}=\tan^2\theta,$$

$$\sec\theta=\frac{1}{\cos\theta},$$

$$\text{cosec}\,\theta=\frac{1}{\sin\theta}\quad(23),$$

$$\cot\theta=\frac{1}{\tan\theta};$$

$$\therefore\ \frac{\sec\theta}{\operatorname{cosec}\theta\,.\,\cot\theta}=\frac{\sin\theta}{\cos\theta}\,.\,\tan\theta$$

$$=\tan^2\theta\quad(26).$$

25. $$\frac{\sec\theta\,.\,\cot\theta-\operatorname{cosec}\theta\,.\,\tan\theta}{\cos\theta-\sin\theta}=\operatorname{cosec}\theta\,.\,\sec\theta,$$

$$\frac{\sec\theta\,.\,\cot\theta-\operatorname{cosec}\theta\,.\,\tan\theta}{\cos\theta-\sin\theta}=\frac{\dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}-\dfrac{1}{\sin\theta}\cdot\dfrac{\sin\theta}{\cos\theta}}{\cos\theta-\sin\theta}\quad(23)\text{ and }(26)$$

$$=\frac{\dfrac{1}{\sin\theta}-\dfrac{1}{\cos\theta}}{\cos\theta-\sin\theta}$$

$$=\frac{\cos\theta-\sin\theta}{\sin\theta\,.\,\cos\theta\,(\cos\theta-\sin\theta)}$$

$$=\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$=\operatorname{cosec}\theta\,.\,\sec\theta.$$

26. $$\sin 2\theta-\operatorname{chd}2\theta\,.\,\cos 2\theta=4\sin\frac{3\theta}{2}\,.\,\sin\theta\,.\,\sin\frac{\theta}{2},$$

$$\sin 2\theta-\operatorname{chd}2\theta\,.\,\cos 2\theta=2\sin\theta\,.\,\cos\theta-2\sin\theta\,.\,\cos 2\theta$$

$$=(\cos\theta-\cos 2\theta)\,.\,2\sin\theta$$

$$=2\sin\left(\frac{2\theta+\theta}{2}\right).\sin\left(\frac{2\theta-\theta}{2}\right).2\sin\theta\ (40)$$

$$=4\sin\frac{3\theta}{2}\,.\,\sin\theta\,.\,\sin\frac{\theta}{2}.$$

27. $$\operatorname{chd}^2 2\theta\left(\frac{1}{1+\cos 2\theta}+\frac{1}{\sin 2\theta}\right)\cdot\left(\frac{1}{1+\cos 2\theta}-\frac{1}{\sin 2\theta}\right)$$

$$=(\tan^2\theta+1)\,.\,(\tan^2\theta-1)$$

$$\operatorname{chd}^2 2\theta\left(\frac{1}{1+\cos 2\theta}+\frac{1}{\sin 2\theta}\right)\cdot\left(\frac{1}{1+\cos 2\theta}-\frac{1}{\sin 2\theta}\right)$$

$$=\left(\frac{\operatorname{chd}2\theta}{1+\cos 2\theta}+\frac{\operatorname{chd}2\theta}{\sin 2\theta}\right)\cdot\left(\frac{\operatorname{chd}2\theta}{1+\cos 2\theta}-\frac{\operatorname{chd}2\theta}{\sin 2\theta}\right)$$

$$= \left(\frac{2\sin\theta}{2\cos^2\theta} + \frac{2\sin\theta}{2\sin\theta\,.\cos\theta}\right).\left(\frac{2\sin\theta}{2\cos^2\theta} - \frac{2\sin\theta}{2\sin\theta\,.\cos\theta}\right)$$

$$= \left(\frac{\sin\theta}{\cos^2\theta} + \frac{1}{\cos\theta}\right).\left(\frac{\sin\theta}{\cos^2\theta} - \frac{1}{\cos\theta}\right)$$

$$= \frac{\sin^2\theta}{\cos^4\theta} - \frac{1}{\cos^2\theta} = \frac{1}{\cos^2\theta}\left(\frac{\sin^2\theta}{\cos^2\theta} - 1\right)$$

$$= \sec^2\theta\,.(\tan^2\theta - 1)$$

$$= (\tan^2\theta + 1)\,.(\tan^2\theta - 1).$$

28. $$\frac{(\operatorname{cosec}\theta + \sec\theta)^2}{\operatorname{cosec}^2\theta + \sec^2\theta} = 1 + \sin 2\theta,$$

$$\operatorname{cosec}\theta + \sec\theta = \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$$

$$= \frac{\cos\theta + \sin\theta}{\sin\theta\,.\cos\theta};$$

$$\therefore\ (\operatorname{cosec}\theta + \sec\theta)^2 = \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\,.\cos\theta}{\sin^2\theta\,.\cos^2\theta}$$

$$= \frac{1 + \sin 2\theta}{\sin^2\theta\,.\cos^2\theta} \quad (23) \text{ and } (29).$$

Also $$\operatorname{cosec}^2\theta + \sec^2\theta = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\,.\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\,.\cos^2\theta};$$

$$\therefore\ \frac{(\operatorname{cosec}\theta + \sec\theta)^2}{\operatorname{cosec}^2\theta + \sec^2\theta} = 1 + \sin 2\theta.$$

29. $$\frac{\cot^2\theta + \cos^2\theta}{\cot^2\theta - \cos^2\theta} = \sec^2\theta + \tan^2\theta,$$

$$\frac{\cot^2\theta + \cos^2\theta}{\cot^2\theta - \cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta\,.\cos^2\theta}{\cos^2\theta - \sin^2\theta\,.\cos^2\theta} \quad (26)$$

$$= \frac{1 + \sin^2\theta}{1 - \sin^2\theta}$$

$$= \frac{1 + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta + \tan^2\theta.$$

30. $$\frac{\tan\theta}{\tan 2\theta - \tan\theta} = \cos 2\theta,$$

$$\frac{\tan 2\theta - \tan\theta}{\tan\theta} = \frac{\tan 2\theta}{\tan\theta} - 1$$

$$= \frac{2}{1 - \tan^2\theta} - 1 \qquad (33)$$

$$= \frac{1 + \tan^2\theta}{1 - \tan^2\theta}$$

$$= \frac{1}{\cos 2\theta} \quad (30);$$

$$\therefore \frac{\tan\theta}{\tan 2\theta - \tan\theta} = \cos 2\theta.$$

31. $$\frac{\tan 2\theta \,.\, \tan\theta}{\tan 2\theta - \tan\theta} = \sin 2\theta,$$

$$\frac{\tan\theta}{\tan 2\theta - \tan\theta} = \cos 2\theta \quad \text{(Prop. 30)};$$

$$\therefore \frac{\tan 2\theta \,.\, \tan\theta}{\tan 2\theta - \tan\theta} = \cos 2\theta \,.\, \tan 2\theta$$

$$= \cos 2\theta \,.\, \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \sin 2\theta.$$

32. $$\frac{\tan 2\theta - \sin 2\theta}{\tan 2\theta + \sin 2\theta} = \tan^2\theta$$

$$\frac{\tan 2\theta - \sin 2\theta}{\tan 2\theta + \sin 2\theta} = \frac{\tan 2\theta \,.\, \cot 2\theta - \sin 2\theta \,.\, \cot 2\theta}{\tan 2\theta \,.\, \cot 2\theta + \sin 2\theta \,.\, \cot 2\theta}$$

$$= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \quad (23) \text{ and } (26)$$

$$= \tan^2\theta \quad (31).$$

33. $$\sec^2\theta = \frac{2 \sec 2\theta}{\sec 2\theta + 1},$$

$$\cos^2\theta = \frac{1}{2} \,.\, (2 \cos^2\theta)$$

$$= \frac{1}{2}(1 + \cos 2\theta) \quad (32)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sec 2\theta}\right)$$

$$= \frac{1}{2}\left(\frac{\sec 2\theta + 1}{\sec 2\theta}\right),$$

$$\text{and } \sec^2\theta = \frac{1}{\cos^2\theta} \quad (23);$$

$$\therefore \sec^2\theta = \frac{2 \sec 2\theta}{\sec 2\theta + 1}.$$

34. $$\operatorname{cosec}^2\theta = \frac{2 \sec 2\theta}{\sec 2\theta - 1},$$

$$\sin^2\theta = \frac{1}{2}(2 \sin^2\theta).$$

$$= \frac{1}{2}(1 - \cos 2\theta) \quad (32)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sec 2\theta}\right)$$

$$= \frac{1}{2}\left(\frac{\sec 2\theta - 1}{\sec 2\theta}\right),$$

$$\text{and } \operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta} \quad (23);$$

$$\therefore\ \operatorname{cosec}^2\theta = \frac{2\sec 2\theta}{\sec 2\theta - 1}.$$

35. $$\frac{\sin(-2\theta) - 2\cos(-\theta)\,.\,\tan(-\theta)}{\operatorname{cosec}(-\theta)\,.\,\sin(-\theta)\,.\,\operatorname{vers}(-\theta)} = 2\sin\theta,$$

$$\frac{\sin(-2\theta) - 2\cos(-\theta)\,.\,\tan(-\theta)}{\operatorname{cosec}(-\theta)\,.\,\sin(-\theta)\,.\,\operatorname{vers}(-\theta)}$$

$$= \frac{-\sin 2\theta - 2\cos\theta\,.\,(-\tan\theta)}{-\operatorname{cosec}\theta\,.\,(-\sin\theta)\,.\,\operatorname{vers}\theta} \quad (10)$$

$$= \frac{2\sin\theta - 2\sin\theta\,.\,\cos\theta}{\sin\theta\,.\,\operatorname{cosec}\theta\,.\,\operatorname{vers}\theta}$$

$$= \frac{2\,.\,(1-\cos\theta)}{\operatorname{cosec}\theta\,.\,(1-\cos\theta)}$$

$$= \frac{2}{\operatorname{cosec}\theta}$$

$$= 2\sin\theta.$$

36. $$\frac{\{\cot(-\theta)+\sec(-\theta)\}\,.\,\{\cot(-\theta)-\sec(-\theta)\}+\tan^2(-\theta)}{\{\cot(-\theta)+\tan(-\theta)\}\,.\,\{\cot(-\theta)-\tan(-\theta)\}+\sec^2(-\theta)} = \cos 2\theta,$$

$$\frac{\{\cot(-\theta)+\sec(-\theta)\}\,.\,\{\cot(-\theta)-\sec(-\theta)\}+\tan^2(-\theta)}{\{\cot(-\theta)+\tan(-\theta)\}\,.\,\{\cot(-\theta)-\tan(-\theta)\}+\sec^2(-\theta)}$$

$$= \frac{\cot^2(-\theta) - \{\sec^2(-\theta) - \tan^2(-\theta)\}}{\cot^2(-\theta) + \{\sec^2(-\theta) - \tan^2(-\theta)\}}$$

$$= \frac{\cot^2\theta - 1}{\cot^2\theta + 1} \quad (10)$$

$$= \frac{\cot\theta - \dfrac{1}{\cot\theta}}{\cot\theta + \dfrac{1}{\cot\theta}}$$

$$= \frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta}$$

$$= \frac{2\cot 2\theta}{2\operatorname{cosec} 2\theta} \quad (27)$$

$$= \cos 2\theta \quad (25).$$

37. $$2\operatorname{cosec}4\theta=\frac{\tan 2\theta+\cot 2\theta+\operatorname{cosec}2\theta}{1+\cos 2\theta},$$

$$\begin{aligned}2\operatorname{cosec}4\theta\,(1+\cos 2\theta)&=2\operatorname{cosec}4\theta+2\operatorname{cosec}4\theta\,.\cos 2\theta\\&=\tan 2\theta+\cot 2\theta+\frac{2\cos 2\theta}{\sin 4\theta}\quad(27)\text{ and }(23)\\&=\tan 2\theta+\cot 2\theta+\frac{1}{\sin 2\theta}\quad(29)\\&=\tan 2\theta+\cot 2\theta+\operatorname{cosec}2\theta;\end{aligned}$$

$$\therefore\ 2\operatorname{cosec}4\theta=\frac{\tan 2\theta+\cot 2\theta+\operatorname{cosec}2\theta}{1+\cos 2\theta}.$$

38. $$\frac{\sec 2\theta(1+\sin 2\theta)+\operatorname{cosec}2\theta\,(1+\cos 2\theta)}{\cot 2\theta+\tan 2\theta+\operatorname{cosec}2\theta}=1+\tan\theta,$$

$$\sec 2\theta\,(1+\sin 2\theta)=\frac{1}{\cos 2\theta}+\frac{\sin 2\theta}{\cos 2\theta}=\sec 2\theta+\tan 2\theta,$$

and $$\operatorname{cosec}2\theta\,(1+\cos 2\theta)=\frac{1}{\sin 2\theta}+\frac{\cos 2\theta}{\sin 2\theta}=\operatorname{cosec}2\theta+\cot 2\theta;$$

$$\begin{aligned}\therefore\ &\sec 2\theta\,(1+\sin 2\theta)+\operatorname{cosec}2\theta\,(1+\cos 2\theta)\\&=\tan 2\theta+\cot 2\theta+\operatorname{cosec}2\,\theta+\sec 2\theta;\end{aligned}$$

$$\begin{aligned}\therefore\ &\frac{\sec 2\theta(1+\sin 2\theta)+\operatorname{cosec}2\theta(1+\cos 2\theta)}{\tan 2\theta+\cot 2\theta+\operatorname{cosec}2\theta}\\&=1+\frac{\sec 2\theta}{\cot 2\theta+\tan 2\theta+\operatorname{cosec}2\theta}\\&=1+\frac{\sec 2\theta}{2\operatorname{cosec}4\theta\,(1+\cos 2\theta)}\quad(\text{Prop. }37).\\&=1+\frac{\sin 4\theta}{2\cos 2\theta\,(1+\cos 2\theta)}\quad(23)\\&=1+\frac{\sin 2\theta}{1+\cos 2\theta}\quad(29)\\&=1+\frac{2\sin\theta\,.\cos\theta}{2\cos^2\theta}\quad(29)\text{ and }(32)\\&=1+\frac{\sin\theta}{\cos\theta}\\&=1+\tan\theta.\end{aligned}$$

39. $1 + \text{chd}\,4\theta = \dfrac{\cos 3\theta + \sin 3\theta}{\cos\theta - \sin\theta}$,

$$\cos 3\theta + \sin 3\theta = 4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta \quad (34)$$

$$= 4(\cos^3\theta - \sin^3\theta) - 3(\cos\theta - \sin\theta);$$

$$\therefore \frac{\cos 3\theta + \sin 3\theta}{\cos\theta - \sin\theta} = 4\left(\frac{\cos^3\theta - \sin^3\theta}{\cos\theta - \sin\theta}\right) - 3$$

$$= 4(\cos^2\theta + \cos\theta \,.\, \sin\theta + \sin^2\theta) - 3$$

$$= 4(\cos^2\theta + \sin^2\theta) + 4\cos\theta \,.\, \sin\theta - 3$$

$$= 1 + 2\sin 2\theta \quad (23) \text{ and } (29)$$

$$= 1 + \text{chd}\,4\theta \quad (25).$$

40. $\dfrac{\cos\left(\dfrac{\theta-\psi}{2}\right)}{\sin\left(\dfrac{\theta+\psi}{2}\right)} \,.\, \cos\phi - \dfrac{\cos\left(\dfrac{\psi-\theta}{2}\right)}{\cos\left(\dfrac{\psi+\theta}{2}\right)} \,.\, \sin\phi = \dfrac{\cos(\theta+\phi) + \cos(\phi+\psi)}{\sin(\theta+\psi)}$,

$$\cos\left(\frac{\theta-\psi}{2}\right) = \cos\frac{\theta}{2} \,.\, \cos\frac{\psi}{2} + \sin\frac{\theta}{2} \,.\, \sin\frac{\psi}{2} = \cos\left(\frac{\psi-\theta}{2}\right) \quad (37);$$

$$\therefore \frac{\cos\left(\dfrac{\theta-\psi}{2}\right)}{\sin\left(\dfrac{\theta+\psi}{2}\right)} \,.\, \cos\phi - \frac{\cos\left(\dfrac{\psi-\theta}{2}\right)}{\cos\left(\dfrac{\psi+\theta}{2}\right)} \,.\, \sin\phi$$

$$= \frac{\cos\left(\dfrac{\theta-\psi}{2}\right)}{\sin\left(\dfrac{\theta+\psi}{2}\right) \,.\, \cos\left(\dfrac{\theta+\psi}{2}\right)} \left\{\cos\phi \,.\, \cos\left(\frac{\theta+\psi}{2}\right) - \sin\phi \,.\, \sin\left(\frac{\theta+\psi}{2}\right)\right\}$$

$$= \frac{2\cos\left(\dfrac{\theta-\psi}{2}\right) \,.\, \cos\left(\dfrac{\theta+\psi}{2} + \phi\right)}{\sin(\theta+\psi)} \quad (29) \text{ and } (37)$$

$$= \frac{2\cos\left\{\dfrac{(\theta+\phi)+(\phi+\psi)}{2}\right\} \,.\, \cos\left\{\dfrac{(\theta+\phi)-(\phi+\psi)}{2}\right\}}{\sin(\theta+\psi)}$$

$$= \frac{\cos(\theta+\phi) + \cos(\phi+\psi)}{\sin(\theta+\psi)} \quad (40).$$

41. $\sin 45^\circ = \sin 75^\circ - \sin 15^\circ$,

$$\sin 75^\circ - \sin 15^\circ = 2 \,.\, \cos\left(\frac{75^\circ + 15^\circ}{2}\right) . \sin\left(\frac{75^\circ - 15^\circ}{2}\right) \quad (40)$$

$$= 2 \cos 45^\circ \,.\, \sin 30^\circ$$

$$= \cos 45^\circ \quad (14)$$

$$= \sin 45^\circ \quad (15)$$

Cor. Hence $\sin 45^\circ = \cos 15^\circ - \sin 15^\circ$.

42. $\dfrac{\tan 225^\circ}{\sec 225^\circ} = \sin 15^\circ - \sin 75^\circ$,

$$\frac{\tan 225^\circ}{\sec 225^\circ} = \sin 225^\circ \quad (25)$$

$$= - \sin 45^\circ \quad (3)$$

$$= \sin 15^\circ - \sin 75^\circ. \quad \text{(Prop. 41)}.$$

43. $\text{chd } 72^\circ = 2 \cos 54^\circ$,

$$\text{chd } 72^\circ = 2 \sin 36^\circ \quad (25)$$

$$= 2 \cos 54^\circ \quad (1).$$

44. $\cos 36^\circ = 2 \cos 27^\circ \,.\, \cos 63^\circ$,

$$\cos 36^\circ = \sin 54^\circ \quad (2)$$

$$= \sin 126^\circ \quad (3)$$

$$= 2 \sin 63^\circ \,.\, \cos 63^\circ \quad (29)$$

$$= 2 \cos 27^\circ \,.\, \cos 63^\circ \quad (1).$$

45. $\text{cosec } 18^\circ - \text{cosec } 54^\circ = \text{cosec } 30^\circ$,

$$\text{cosec } 18^\circ - \text{cosec } 54^\circ = \frac{1}{\sin 18^\circ} - \frac{1}{\sin 54^\circ}$$

$$= \frac{\sin 54^\circ - \sin 18^\circ}{\sin 54^\circ \,.\, \sin 18^\circ}$$

$$= \frac{2\cos\left(\frac{54^\circ + 18^\circ}{2}\right) . \sin\left(\frac{54^\circ - 18^\circ}{2}\right)}{\cos 36^\circ . \sin 18^\circ} \quad (40)$$

$$= \frac{2\cos 36^\circ . \sin 18^\circ}{\cos 36^\circ . \sin 18^\circ}$$

$$= 2$$

$$= \operatorname{cosec} 30^\circ \quad (13).$$

46. $\sec 72^\circ - \sec 36^\circ = \sec 60^\circ,$

$$\sec 72^\circ - \sec 36^\circ = \frac{1}{\cos 72^\circ} - \frac{1}{\cos 36^\circ}$$

$$= \frac{\cos 36^\circ - \cos 72^\circ}{\cos 36^\circ . \cos 72^\circ}$$

$$= \frac{2\sin\left(\frac{72^\circ + 36^\circ}{2}\right) . \sin\left(\frac{72^\circ - 36^\circ}{2}\right)}{\sin 54^\circ . \sin 18^\circ} \quad (40)\ (1) \text{ and } (2)$$

$$= \frac{2\sin 54^\circ . \sin 18^\circ}{\sin 54^\circ . \sin 18^\circ}$$

$$= 2$$

$$= \sec 60^\circ \quad (13).$$

47. $\operatorname{chd} 225^\circ = \operatorname{chd} 135^\circ,$

$$\operatorname{chd} 225^\circ = 2\sin 112^\circ . 30' \quad (25)$$

$$= 2\sin 67^\circ . 30' \quad (3)$$

$$= \operatorname{chd} 135^\circ \quad (25).$$

48. $\sin^4 67^\circ . 30' - \sin^4 22^\circ . 30' = \sin 45^\circ$

$$\sin^4 67^\circ . 30' - \sin^4 22^\circ . 30' = \sin^4\left(\frac{3\pi}{8}\right) - \sin^4\left(\frac{\pi}{8}\right)$$

$$= \cos^4\left(\frac{\pi}{2} - \frac{3\pi}{8}\right) - \sin^4\frac{\pi}{8} \quad (1)$$

$$= \cos^4\frac{\pi}{8} - \sin^4\frac{\pi}{8}$$

$$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) . \left(\cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8}\right)$$

$$= \cos\frac{\pi}{4} \quad (23) \text{ and } (30)$$

$$= \sin 45^0 \quad (15).$$

49. sec 45°. vers 45° = chd 270° − chd 60°,

$$\sec 45^0 . \operatorname{vers} 45^0 = \frac{1-\cos 45^0}{\cos 45^0}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \quad (15)$$

$$= \sqrt{2} - 1$$

$$= \operatorname{chd} 270^0 - \operatorname{chd} 60^0 \quad (16) \text{ and } (12).$$

50. sec 225° = cos 135° − cos 45°,

$$\cos 135^0 - \cos 45^0 = -2\cos 45^0 \quad (4)$$

$$= -\sqrt{2} \quad (15)$$

$$= -\sec 45^0 \quad (16)$$

$$= \sec 225^0 \quad (7).$$

51. cos 15° + sin 15° = ½ (chd 300° . chd 270° . chd 240°),

$$\cos 15^0 + \sin 15^0 = \sin 75^0 + \sin 15^0 \quad (2)$$

$$= 2\sin\left(\frac{75^0+15^0}{2}\right) . \cos\left(\frac{75^0-15^0}{2}\right) \quad (40)$$

$$= 2\sin 45^0 . \cos 30^0$$

$$= 2\sin 45^0 . \sin 60^0 \quad (2)$$

$$= \tfrac{1}{2}(2\sin 30^0 . 2\sin 45^0 . 2\sin 60^0) \quad (14)$$

$$= \tfrac{1}{2}(2\sin 150^0 . 2\sin 135^0 . 2\sin 120^0) \quad (3)$$

$$= \tfrac{1}{2}(\operatorname{chd} 300^0 . \operatorname{chd} 270^0 . \operatorname{chd} 240^0) \quad (25).$$

52. $\text{chd } 252^\circ = (\text{chd } 108^\circ + \text{chd } 60^\circ) \,.\, (\text{chd } 108^\circ - \text{chd } 60^\circ)$

$$\begin{aligned}\text{chd } 252^\circ &= 2 \sin 126^\circ \quad (25)\\ &= 2 \sin 54^\circ \quad (3)\\ &= \frac{\sqrt{5}+1}{2} \quad (22),\end{aligned}$$

but

$$\begin{aligned}(\text{chd } 108^\circ + \text{chd } 60^\circ) \,.\, (\text{chd } 108^\circ - \text{chd } 60^\circ) &= \text{chd}^2 108^\circ - \text{chd}^2 60\\ &= 4 \sin^2 54^\circ - 4 \sin^2 30^\circ \quad (25)\\ &= \frac{(\sqrt{5}+1)^2}{4} - 1 \quad (22) \text{ and } (14)\\ &= \frac{3+\sqrt{5}}{2} - 1\\ &= \frac{\sqrt{5}+1}{2};\end{aligned}$$

$\therefore\ \text{chd } 252^\circ = (\text{chd } 108^\circ + \text{chd } 60^\circ)\, (\text{chd } 108^\circ - \text{chd } 60^\circ).$

53. $\sin 108^\circ = (\sin 81^\circ + \sin 9^\circ) \,.\, (\sin 81^\circ - \sin 9^\circ),$

$$\begin{aligned}\sin 108^\circ &= \sin 72^\circ \quad (3)\\ &= \sin 90^\circ \,.\, \sin 72^\circ \quad (12)\\ &= \sin (81^\circ + 9^\circ) \,.\, \sin (81^\circ - 9^\circ)\\ &= \sin^2 81^\circ - \sin^2 9^\circ \quad (41)\\ &= (\sin 81^\circ + \sin 9^\circ) \,.\, (\sin 81^\circ - \sin 9^\circ).\end{aligned}$$

54. $\sin^2 33^\circ + \sin^2 27^\circ = \sin^2 45^\circ + \sin^2 3^\circ,$

$$\begin{aligned}\sin^2 33^\circ - \sin^2 3^\circ &= \sin (33+3)^\circ \,.\, \sin (33-3)^\circ \quad (41)\\ &= \sin 36^\circ \,.\, \sin 30^\circ\\ &= \tfrac{1}{2} \sin 36^\circ \quad (14)\\ &= \sin 18^\circ \,.\, \cos 18^\circ \quad (29)\\ &= \sin 18^\circ \,.\, \sin 72^\circ \quad (2)\\ &= \sin (45+27)^\circ \,.\, \sin (45-27)^\circ\\ &= \sin^2 45^\circ - \sin^2 27^\circ; \quad (41)\end{aligned}$$

$\therefore\ \sin^2 33^\circ + \sin^2 27^\circ = \sin^2 45^\circ + \sin^2 3^\circ.$

55. $\sin^2 63^\circ - \sin^2 27^\circ = \sin 144^\circ$,

$$\begin{aligned}\sin^2 63^\circ - \sin^2 27^\circ &= \cos^2 27^\circ - \sin^2 27^\circ \quad (1)\\ &= \cos 54^\circ \quad (30)\\ &= \sin 36^\circ \quad (2)\\ &= \sin 144^\circ \quad (3).\end{aligned}$$

56. $\sin^2 54^\circ - \sin^2 18^\circ = \cos 54^\circ . \cos 18^\circ$.

$$\begin{aligned}\sin^2 54^\circ - \sin^2 18^\circ &= \cos^2 36^\circ - \sin^2 18^\circ \quad (1)\\ &= \cos(36^\circ + 18^\circ) . \cos(36^\circ - 18^\circ) \quad (41)\\ &= \cos 54^\circ . \cos 18^\circ.\end{aligned}$$

57. $\dfrac{\cos 3^\circ - \cos 33^\circ}{\sin 3^\circ + \sin 33^\circ} = \tan 15^\circ$.

$$\begin{aligned}\frac{\cos 3^\circ - \cos 33^\circ}{\sin 3^\circ + \sin 33^\circ} &= \frac{2 \sin \left(\frac{33+3}{2}\right)^\circ . \sin \left(\frac{33-3}{2}\right)^\circ}{2 \sin \left(\frac{33+3}{2}\right)^\circ . \cos \left(\frac{33-3}{2}\right)^\circ} \quad (40)\\ &= \frac{\sin 15^\circ}{\cos 15^\circ}\\ &= \tan 15^\circ.\end{aligned}$$

58. $\dfrac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ} = \cot 72^\circ$,

$$\begin{aligned}\frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ} &= \frac{2 \sin \left(\frac{33+3}{2}\right)^\circ . \cos \left(\frac{33-3}{2}\right)^\circ}{2 \cos \left(\frac{33+3}{2}\right)^\circ . \cos \left(\frac{33-3}{2}\right)^\circ} \quad (40)\\ &= \frac{\sin 18^\circ}{\cos 18^\circ}\\ &= \tan 18^\circ\\ &= \cot 72^\circ.\end{aligned}$$

59. $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan 54^\circ,$

$$\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \frac{1 + \dfrac{\sin 9^\circ}{\cos 9^\circ}}{1 - \dfrac{\sin 9^\circ}{\cos 9^\circ}}$$

$$= \frac{1 + \tan 9^\circ}{1 - \tan 9^\circ}$$

$$= \frac{\tan 45^\circ + \tan 9^\circ}{1 - \tan 45^\circ . \tan 9^\circ} \quad (12)$$

$$= \tan (45 + 9)^\circ \quad (38)$$

$$= \tan 54^\circ.$$

60. $\dfrac{\cos 27^\circ - \sin 27^\circ}{\cos 27^\circ + \sin 27^\circ} = \tan 18^\circ,$

$$\frac{\cos 27^\circ - \sin 27^\circ}{\cos 27^\circ + \sin 27^\circ} = \frac{1 - \dfrac{\sin 27^\circ}{\cos 27^\circ}}{1 + \dfrac{\sin 27^\circ}{\cos 27^\circ}}$$

$$= \frac{1 - \tan 27^\circ}{1 + \tan 27^\circ}$$

$$= \frac{\tan 45^\circ - \tan 27^\circ}{1 + \tan 45^\circ . \tan 27^\circ} \quad (12)$$

$$= \tan (45 - 27)^\circ \quad (38)$$

$$= \tan 18^\circ.$$

61. $\operatorname{cosec} 100^\circ = \dfrac{\sec^2 50^\circ - 4 \sec 10^\circ . \tan 10^\circ}{2 \tan 40^\circ},$

$$\sec^2 50^\circ - 2 \operatorname{cosec} 100^\circ . \tan 40^\circ$$

$$= \tan^2 50^\circ + 1 - \tan 40^\circ (\cot 50^\circ + \tan 50^\circ) \quad (23) \text{ and } (27)$$

$$= \tan^2 50^\circ + \tan 50^\circ . \cot 50^\circ - \tan 40^\circ (\cot 50^\circ + \tan 50^\circ) \quad (23)$$

$$= \tan 50^\circ (\cot 50^\circ + \tan 50^\circ) - \tan 40^\circ (\cot 50^\circ + \tan 50^\circ)$$

$$= (\tan 50^\circ - \tan 40^\circ) . (\cot 50^\circ + \tan 50^\circ)$$

$$= \{\tan (45 + 5)^\circ - \tan (45 - 5)^\circ\} . \{\cot (45 + 5)^\circ + \tan (45 + 5)^\circ\}$$

$$= \left(\frac{\tan 45^\circ + \tan 5^\circ}{1 - \tan 45^\circ \,.\, \tan 5^\circ} - \frac{\tan 45^\circ - \tan 5^\circ}{1 + \tan 45^\circ \,.\, \tan 5^\circ}\right)$$

$$\left(\frac{1 - \tan 45^\circ \,.\, \tan 5^\circ}{\tan 45^\circ + \tan 5^\circ} + \frac{\tan 45^\circ + \tan 5^\circ}{1 - \tan 45^\circ \,.\, \tan 5^\circ}\right) \quad (38)$$

$$= \left(\frac{1 + \tan 5^\circ}{1 - \tan 5^\circ} - \frac{1 - \tan 5^\circ}{1 + \tan 5^\circ}\right) . \left(\frac{1 - \tan 5^\circ}{1 + \tan 5^\circ} + \frac{1 + \tan 5^\circ}{1 - \tan 5^\circ}\right) \quad (12)$$

$$= \left\{\frac{(1 + \tan 5^\circ)^2 - (1 - \tan 5^\circ)^2}{1 - \tan^2 5^\circ}\right\} \left\{\frac{(1 - \tan 5^\circ)^2 + (1 + \tan 5^\circ)^2}{1 - \tan^2 5^\circ}\right\}$$

$$= 2 \,.\, \left(\frac{2 \tan 5^\circ}{1 - \tan^2 5^\circ}\right) 2 \,.\, \left(\frac{1 + \tan^2 5^\circ}{1 - \tan^2 5^\circ}\right)$$

$$= 4 \tan 10^\circ \,.\, \sec 10^\circ \qquad (33) \text{ and } (30);$$

$$\therefore\ \sec^2 50^\circ - 4 \tan 10^\circ \,.\, \sec 10^\circ = 2 \operatorname{cosec} 100^\circ \,.\, \tan 40^\circ;$$

$$\therefore\ \operatorname{cosec} 100^\circ = \frac{\sec^2 50^\circ - 4 \tan 10^\circ \,.\, \sec 10^\circ}{2 \tan 40^\circ}.$$

62. $\operatorname{cosec} 75^\circ \,.\, \operatorname{cosec} 15^\circ = 4$,

$$\operatorname{cosec} 75^\circ \,.\, \operatorname{cosec} 15^\circ = \frac{1}{\sin 75^\circ} \,.\, \frac{1}{\sin 15^\circ}$$

$$= \frac{2\sqrt{2}}{\sqrt{3} + 1} \,.\, \frac{2\sqrt{2}}{\sqrt{3} - 1} \quad (21)$$

$$= \frac{8}{2}$$

$$= 4.$$

Cor. Hence $\sec 15^\circ \,.\, \operatorname{cosec} 15^\circ = 4$.

Each of the expressions in this chapter is calculated to Radius (1), and might be proved in a similar manner if (r) were assumed as Radius; or it might at once be transferred from Radius (1) to Radius (r) by the rule laid down in the Elementary Treatises.

CHAPTER II.

PROGRESSIONS.

63. Vers 45° is an *Arithmetic* mean between sec 60° and sec 225°,

$$2 \text{ vers } 45^\circ = 2 - 2\cos 45^\circ \qquad (24)$$

$$= 2 - \sqrt{2} \qquad (15)$$

$$= \sec 60^\circ - \sec 45^\circ \qquad (13) \text{ and } (16)$$

$$= \sec 60^\circ + \sec 225^\circ \qquad (7);$$

$$\therefore \text{ vers } 45^\circ = \frac{\sec 60^\circ + \sec 225^\circ}{2}.$$

That is, vers 45° is an arithmetic mean between sec 60° and sec 225°.

64. cosec 150° is a *Geometric* mean between cosec 105° and cosec 165°.

$$\operatorname{cosec}^2 150^\circ = \operatorname{cosec}^2 30^\circ \quad (8)$$

$$= 4 \qquad (13)$$

$$= \operatorname{cosec} 75^\circ . \operatorname{cosec} 15^\circ \quad (\text{Prop. } 62)$$

$$= \operatorname{cosec} 105 . \operatorname{cosec} 165^\circ \quad (8);$$

$$\therefore \operatorname{cosec} 150^\circ = \sqrt{\operatorname{cosec} 105^\circ . \operatorname{cosec} 165^\circ}.$$

That is, cosec 150° is a Geometric mean between cosec 105° and cosec 165°.

65. cos 60° is a *Geometric* mean between cos 36° and cos 72°.

$$\cos^2 60^\circ = \frac{1}{4} \quad (14)$$

$$= \frac{4}{16}$$

$$= \frac{5-1}{16}$$

$$= \left(\frac{\sqrt{5}+1}{4}\right) . \left(\frac{\sqrt{5}-1}{4}\right)$$

$$= \cos 36^\circ . \cos 72^\circ \quad (22).$$

That is, $\cos 60^\circ$ is a Geometric mean between $\cos 36^\circ$ and $\cos 72^\circ$.

66. $\sin \frac{\pi}{4}$ is an *Harmonic* mean between $\tan \frac{\pi}{8}$ and $\cot \frac{\pi}{8}$.

$$\sin \frac{\pi}{4} = \frac{2 \sin \frac{\pi}{8} . \cos \frac{\pi}{8}}{\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8}} \quad (29) \text{ and } (23)$$

$$= \frac{2}{\dfrac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8}} + \dfrac{\cos \frac{\pi}{8}}{\sin \frac{\pi}{8}}}$$

$$= \frac{2 \tan \frac{\pi}{8} . \cot \frac{\pi}{8}}{\tan \frac{\pi}{8} + \cot \frac{\pi}{8}} \quad (23) \text{ and } (26).$$

That is, $\sin \frac{\pi}{4}$ is an *Harmonic* mean between $\tan \frac{\pi}{8}$ and $\cot \frac{\pi}{8}$.

67. If $\text{chd}^2 \theta$, $\text{chd}^2 \phi$ and $\text{chd}^2 \psi$ be in *Arithmetic* Progression; then $\sec \theta$, $\sec \phi$ and $\sec \psi$ will be in *Harmonic* Progression.

$$\sec \phi = \frac{1}{\cos \phi}$$

$$= \frac{1}{1 - \text{vers}\, \phi}$$

$$= \frac{2}{2 - 2\, \text{vers}\, \phi}$$

$$= \frac{2}{2 - \text{chd}^2 \phi} \quad (25)$$

$$=\frac{4}{4-2\,\text{chd}^2\phi}$$

$$=\frac{4}{4-(\text{chd}^2\theta+\text{chd}^2\psi)}\quad\text{by hypothesis}$$

$$=\frac{4}{(2-\text{chd}^2\theta)+(2-\text{chd}^2\psi)}$$

$$=\frac{4}{2\cos\theta+2\cos\psi}\quad(25)$$

$$=\frac{2}{\cos\theta+\cos\psi}$$

$$=\frac{2\sec\theta\,.\,\sec\psi}{\cos\theta\,.\,\sec\theta\,.\,\sec\psi+\cos\psi\,.\,\sec\psi\,.\,\sec\theta}$$

$$=\frac{2\sec\theta\,.\,\sec\psi}{\sec\psi+\sec\theta}\quad(23);$$

$\therefore$ $\sec\theta$, $\sec\phi$, $\sec\psi$ are in *Harmonic* Progression.

68. The value of $\theta+\phi+\psi$ is $\frac{3\pi}{8}$ where θ, ϕ, ψ are in *Arithmetic* Progression, shew that

$$\frac{\text{cosec}^2\theta-\sec^2\theta}{\text{cosec}^2\theta+\sec^2\theta}=\sin 2\psi,$$

$$\frac{\text{cosec}^2\theta-\sec^2\theta}{\text{cosec}^2\theta+\sec^2\theta}=\frac{\frac{1}{\sin^2\theta}-\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}}$$

$$=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$=\cos 2\theta\quad(23)\text{ and }(30).$$

Now $\frac{3\pi}{8}=\theta+\psi+\phi=\theta+\psi+\frac{\theta+\psi}{2}$ by hypothesis

$$=\frac{3}{2}\,.\,(\theta+\psi);$$

$$\therefore\ \theta+\psi=\frac{\pi}{4}\text{ and }\theta=\frac{\pi}{4}-\psi;\quad\therefore\ 2\theta=\frac{\pi}{2}-2\psi,$$

and $\cos 2\theta=\sin 2\psi$;

$$\text{consequently } \frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta} = \sin 2\psi.$$

69. $\theta + \phi + \psi = \frac{\pi}{2}$ and $\cos 2\theta$, $\cos 2\phi$, $\cos 2\psi$ are in *Arithmetic* Progression, shew that

$\tan\theta$, $\tan\phi$, $\tan\psi$ are also in *Arithmetic* Progression.

$$\cos 2\theta + \cos 2\psi = 2\cos 2\phi \text{ by hypothesis,}$$

$$\text{or } \cos 2\psi - \cos 2\phi = \cos 2\phi - \cos 2\theta;$$

$$\therefore\ 2\sin(\phi+\psi)\,.\,\sin(\phi-\psi) = 2\sin(\theta+\phi)\,.\,\sin(\theta-\phi) \quad (40)$$

$$\sin(\phi-\psi)\,.\,\cos\theta = \sin(\theta-\phi)\,.\,\cos\psi \quad (2);$$

$$\therefore\ \frac{\sin(\phi-\psi)}{\cos\phi\,.\,\cos\psi} = \frac{\sin(\theta-\phi)}{\cos\theta\,.\,\cos\phi};$$

$$\therefore\ \frac{\sin\phi\,.\,\cos\psi - \cos\phi\,.\,\sin\psi}{\cos\phi\,.\,\cos\psi} = \frac{\sin\theta\,.\,\cos\phi - \cos\theta\,.\,\sin\phi}{\cos\theta\,.\,\cos\phi} \quad (36)$$

$$\text{or } \frac{\sin\phi}{\cos\phi} - \frac{\sin\psi}{\cos\psi} = \frac{\sin\theta}{\cos\theta} - \frac{\sin\phi}{\cos\phi}$$

$$\text{or } \tan\phi - \tan\psi = \tan\theta - \tan\phi;$$

$$\therefore\ \tan\theta + \tan\psi = 2\tan\phi,$$

or $\tan\theta$, $\tan\phi$, $\tan\psi$ are in *Arithmetic* Progression.

70. The sum of three arcs is 135; their tangents are in *Geometric* Progression, and the sum of the tangents of the extremes is 4. What are the arcs?

Let the tangents of θ, ϕ, ψ be respectively denoted by $\frac{x}{\sqrt{y}}$, x and $x\,.\sqrt{y}$.

$$\text{Then } \frac{x}{\sqrt{y}} + x\sqrt{y} = 4 = \tan\theta + \tan\psi \text{ by hypothesis,}$$

$$\text{and } \tan\theta\,.\,\tan\psi = \frac{x}{\sqrt{y}}\,.\,x\,.\sqrt{y} = x^2.$$

Also $\theta + \psi = 135 - \phi$;

$$\therefore\ \tan(\theta+\psi) = \tan(135-\phi),$$

$$\text{or } \frac{\tan\theta + \tan\psi}{1 - \tan\theta . \tan\psi} = \frac{\tan 135 - \tan\phi}{1 + \tan 135 . \tan\phi} \quad (38);$$

$$\therefore \frac{4}{1 - x^2} = \frac{-1 - x}{1 - x} \quad (12)$$

$$= \frac{x + 1}{x - 1};$$

$$\therefore 4x - 4 = x + 1 - x^3 - x^2;$$

$$\therefore x^3 + x^2 + 3x - 5 = 0,$$

$$\text{or } (x^3 - 1) + (x^2 - 1) + 3(x - 1) = 0,$$

$$\text{or } \{(x^2 + x + 1) + (x + 1) + 3\} . (x - 1) = 0.$$

Taking the quadratic factor $x^2 + 2x + 5 = 0$, we get two imaginary roots; but taking the simple factor $x - 1 = 0$, we have $x = 1$;

$$\therefore 4 = \frac{1}{\sqrt{y}} + \sqrt{y};$$

$$\therefore y - 4\sqrt{y} = -1;$$

$$\therefore y - 4\sqrt{y} + 4 = 3;$$

$$\therefore \sqrt{y} - 2 = \pm\sqrt{3};$$

$$\therefore \sqrt{y} = 2 \pm \sqrt{3};$$

$$\therefore \tan\theta = \frac{1}{2 \pm \sqrt{3}} = 2 \mp \sqrt{3}; \quad \text{and } \therefore \theta = 15 \text{ and } 75 \quad (21).$$

$$\tan\psi = 2 \pm \sqrt{3}; \quad \text{and } \therefore \psi = 75 \text{ and } 15.$$

$$\tan\phi = 1; \quad \text{and } \therefore \phi = 45 \quad (12).$$

71. If $S_1 = \sin\theta + \sin^2\theta + \sin^3\theta + \ldots\ldots$

and $S_2 = \sin\theta - \sin^2\theta + \sin^3\theta - \ldots\ldots$

Then $\tan\theta$ is a *Geometric* Mean between S_1 and S_2,

$\sin\theta$ .. an *Harmonic* .... S_1 and S_2,

$\operatorname{cosec}\theta$ ..... *Arithmetic* .... S_1^{-1} and S_2^{-1},

$$S_1 = \sin\theta + \sin^2\theta + \sin^3\theta + \ldots\ldots = \frac{\sin\theta}{1-\sin\theta},$$

$$S_2 = \sin\theta - \sin^2\theta + \sin^3\theta - \ldots\ldots = \frac{\sin\theta}{1+\sin\theta};$$ HIND'S *Algebra,* 270.

$$\therefore\ S_1 . S_2 = \frac{\sin^2\theta}{1-\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta;$$

$\therefore\ \sqrt{S_1 . S_2} = \tan\theta$; which is $\therefore$ a *Geometric* Mean between S_1 and S_2.

$$\text{Again }\ S_1 + S_2 = \frac{\sin\theta}{1-\sin\theta} + \frac{\sin\theta}{1+\sin\theta} = \frac{2\sin\theta}{1-\sin^2\theta} = \frac{2\sin\theta}{\cos^2\theta};$$

$$\text{and }\ 2S_1 . S_2 = \frac{2\sin^2\theta}{\cos^2\theta};$$

$\therefore\ \dfrac{2S_1 . S_2}{S_1 + S_2} = \sin\theta$; which is $\therefore$ an *Harmonic* Mean between S_1 and S_2.

$$\text{Also }\ S_1^{-1} + S_2^{-1} = \frac{1-\sin\theta}{\sin\theta} + \frac{1+\sin\theta}{\sin\theta} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta;$$

$\therefore\ \dfrac{S_1^{-1} + S_2^{-1}}{2} = \operatorname{cosec}\theta$; which is $\therefore$ an *Arithmetic* Mean between S_1^{-1} and S_2^{-1}.

72. In a plane triangle

$$\sin^2\frac{B}{2} = \cos\left(\frac{A+B}{2}\right) . \cos\left(\frac{A+C}{2}\right) . \cos\left(\frac{B+C}{2}\right);$$

shew that chd A, chd B, chd C are in *Geometric* Progression.

$$\sin^2\frac{B}{2} = \cos\left(\frac{A+B}{2}\right) . \cos\left(\frac{A+C}{2}\right) . \cos\left(\frac{B+C}{2}\right)$$

$$= \sin\frac{C}{2} . \sin\frac{B}{2} . \sin\frac{A}{2} \quad (1);$$

$$\therefore\ \sin^2\frac{B}{2} = \sin\frac{C}{2} . \sin\frac{A}{2};$$

$$\therefore\ 4\sin^2\frac{B}{2} = 2\sin\frac{C}{2} . 2\sin\frac{A}{2},$$

$$\text{or chd}^2 B = \text{chd}\, C . \text{chd}\, A \quad (25);$$

$$\therefore\ \text{chd}\, B = \sqrt{(\text{chd}\, C . \text{chd}\, A)}.$$

That is, chd A, chd B, chd C are in *Geometric* Progression.

73. In a plane triangle $\tan A \,.\, \tan C = 3$, shew that $\cot A$, $\cot B$, and $\cot C$ are in *Harmonic* Progression.

$$3 = \tan A \,.\, \tan C;$$

$$\therefore\ 3 \tan B = \tan A \,.\, \tan B \,.\, \tan C = \tan A + \tan B + \tan C \quad (47);$$

$$\therefore\ 2 \tan B = \tan A + \tan C;$$

$\therefore$ $\tan A$, $\tan B$, $\tan C$ are in *Arithmetic* Progression;

and $\therefore$ $\cot A$, $\cot B$, $\cot C$ their *reciprocals*, will be in *Harmonic* Progression. HIND's *Algebra*, 277.

74. $\text{vers}\, A$, $\text{vers}\, B$, $\text{vers}\, C$ are in *Arithmetic* Progression.

Also

$$\tan\left(\frac{A+C}{2}\right) - \cot\left(\frac{A+C}{2}\right) = \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{A+C}{2}\right)}.$$

What is the value of $A + B + C$?

$$\cos\left(\frac{A-C}{2}\right) = \tan\left(\frac{A+C}{2}\right) . \sin\left(\frac{A+C}{2}\right) - \cot\left(\frac{A+C}{2}\right) . \sin\left(\frac{A+C}{2}\right)$$

$$= \frac{\sin^2\left(\frac{A+C}{2}\right)}{\cos\left(\frac{A+C}{2}\right)} - \cos\left(\frac{A+C}{2}\right) \quad (26);$$

$$\therefore\ \cos\left(\frac{A+C}{2}\right) . \cos\left(\frac{A-C}{2}\right) = \sin^2\left(\frac{A+C}{2}\right) - \cos^2\left(\frac{A+C}{2}\right)$$

$$= -\cos(A+C) \quad (30)$$

$$= \cos\{\pi - (A+C)\} \quad (4);$$

$$\therefore\ 2\cos\{\pi - (A+C)\} = 2\cos\left(\frac{A+C}{2}\right) . \cos\left(\frac{A-C}{2}\right)$$

$$= \cos A + \cos C \quad (40)$$

$$= 2 - (\text{vers}\, A + \text{vers}\, C) \quad (24)$$

$$= 2 - 2\,\text{vers}\, B, \text{ by hypothesis};$$

$$\therefore \cos\{\pi - (A + C)\} = 1 - \text{vers } B$$
$$= \cos B;$$
$$\therefore \pi - (A + C) = B;$$
$$\therefore A + B + C = \pi.$$

75. A, B, C the angles of a plane triangle are in *Arithmetic* Progression, and

$$\frac{\sin A}{\sin C} = \frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}.$$

Shew that (x) is the common difference.

$$3B = 2B + B = A + C + B, \text{ by hypothesis}$$
$$= \pi;$$
$$\therefore B = \frac{\pi}{3}; \quad \therefore \frac{A + C}{2} = B = \frac{\pi}{3}.$$

Now $\sqrt{3} + \tan x : \sqrt{3} - \tan x :: \sin A : \sin C$;

$$\therefore \sqrt{3} : \tan x :: \sin A + \sin C : \sin A - \sin C.$$

HIND's *Algebra*, 241, case 6.

$$:: \tan\left(\frac{A + C}{2}\right) : \tan\left(\frac{A - C}{2}\right) \quad (44)$$
$$:: \tan\frac{\pi}{3} : \tan\left(\frac{A - C}{2}\right).$$

But $\tan\frac{\pi}{3} = \sqrt{3}$ (19);

$$\therefore \tan x = \tan\frac{A - C}{2};$$
$$\therefore x = \frac{A}{2} - \frac{C}{2},$$
$$\text{and } \frac{\pi}{3} = \frac{A}{2} + \frac{C}{2};$$

$$\therefore \frac{\pi}{3} + x = A, \text{ and } \frac{\pi}{3} - x = C;$$

$$\therefore \text{the angles are } \frac{\pi}{3} + x, \ \frac{\pi}{3} \text{ and } \frac{\pi}{3} - x,$$

an Arithmetic Progression whose common difference is (x).

76. *A, B, C* are in Arithmetic Proportion, and *AD* which is perpendicular to *BC* divides it in *D*, so that $BD : DC :: 1 : 3$.

Find the values of *A, B* and *C*.

$$B = 60. \qquad \text{(Prop. 75)}.$$

Now $BD : AD :: 1 : \tan B$

$AD : DC :: \tan C : 1$;

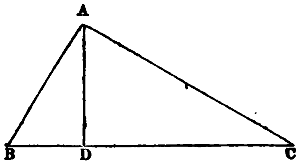
$\therefore BD : DC :: \tan C : \tan B$. (Euclid, Book v. Prop. 23),

or $1 : 3 :: \tan C : \sqrt{3}$ (19) and by hypothesis;

$$\therefore \tan C = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \tan 30 \quad (20);$$

$$\therefore C = 30;$$

$$\therefore A = 180 - (B + C)$$
$$= 180 - 90$$
$$= 90.$$



CHAPTER III.

RIGHT-ANGLED TRIANGLES.

77. $$\frac{\text{Chd}\,(A-B)}{\text{chd}\,(A+B)} = \sin A - \cos A,$$

$$A + B = \pi - C = \pi - \frac{\pi}{2} = \frac{\pi}{2}; \quad \therefore \frac{A+B}{2} = \frac{\pi}{4},$$

$$\text{and } \text{chd}\,(A+B) = \sqrt{2} \quad (16)$$

$$= \frac{1}{\cos\frac{\pi}{4}} \quad (15)$$

$$= \frac{1}{\cos\left(\frac{A+B}{2}\right)},$$

$$\text{and } \text{chd}\,(A-B) = 2\sin\left(\frac{A-B}{2}\right) \quad (25);$$

$$\therefore \frac{\text{chd}\,(A-B)}{\text{chd}\,(A+B)} = 2\sin\left(\frac{A-B}{2}\right).\cos\left(\frac{A+B}{2}\right)$$

$$= \sin A - \sin B \quad (40)$$

$$= \sin A - \cos A \quad (2).$$

78. $$\frac{\text{chd}^2\,2A\,.\,\sin 2B}{\text{chd}^2\,2B\,.\,\sin 2A} = \frac{\tan A}{\tan B},$$

$$B = \frac{\pi}{2} - A; \quad \therefore 2B = \pi - 2A, \quad \text{and } \sin 2B = \sin 2A \quad (3);$$

$$\therefore \frac{\text{chd}^2\,2A\,.\,\sin 2B}{\text{chd}^2\,2B\,.\,\sin 2A} = \frac{4\sin^2 A}{4\sin^2 B}$$

$$= \frac{\sin^2 A}{\cos^2 A}$$

$$=\frac{\dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\sin A}}$$

$$=\frac{\tan A}{\cot A} \quad (26)$$

$$=\frac{\tan A}{\tan B} \quad (5).$$

79. $$\frac{\cos 2B-\cos 2A}{\sin 2A}=\tan A-\tan B,$$

$$\frac{\cos 2B-\cos 2A}{\sin 2A}=\frac{2\sin(A+B).\sin(A-B)}{2\sin A.\cos A} \quad (40) \text{ and } (29)$$

$$=\frac{\sin^2 A-\sin^2 B}{\sin A.\cos A} \quad (41)$$

$$=\frac{\sin^2 A-\cos^2 A}{\sin A.\cos A}$$

$$=\frac{\sin A}{\cos A}-\frac{\cos A}{\sin A}$$

$$=\tan A-\cot A \quad (26)$$

$$=\tan A-\tan B \quad (5).$$

80. $$\sec^2 2A-\tan^2 2B=\text{chd}^2 A\left(1-\tfrac{1}{2}\text{vers}A\right)+\text{chd}^2 B\left(1-\tfrac{1}{2}\text{vers}B\right),$$

$$\sec^2 2A-\tan^2 2B=\sec^2 2A-\tan^2(\pi-2A)$$

$$=\sec^2 2A-\tan^2 2A \quad (5)$$

$$=1 \quad (23).$$

$$\text{chd}^2 A\left(1-\tfrac{1}{2}\text{vers}A\right)=\tfrac{1}{2}\text{chd}^2 A\,(2-\text{vers}A)$$

$$=\text{vers}A\{2-(1-\cos A)\} \quad (24) \text{ and } (25)$$

$$=(1-\cos A).(1+\cos A)$$

$$=1-\cos^2 A$$

$$=\sin^2 A.$$

Similarly $\text{chd}^2 B\left(1-\frac{1}{2}\text{vers}\,B\right)=\sin^2 B$;

$$\therefore\ \text{chd}^2 A\left(1-\tfrac{1}{2}\text{vers}\,A\right)+\text{chd}^2 B\left(1-\tfrac{1}{2}\text{vers}\,B\right)=\sin^2 A+\sin^2 B$$
$$=\sin^2 A+\cos^2 A$$
$$=1$$
$$=\sec^2 2A-\tan^2 2B.$$

81. If $\cos A\,.\,\cos B\,.\,\sin C=\dfrac{\sin A+\sin B}{\sec A+\sec B}$, the triangle will be *right-angled.*

$$\text{Here }\sin C=\frac{\sin A+\sin B}{\sec A\,.\,\cos A\,.\,\cos B+\sec B\,.\,\cos B\,.\,\cos A}$$
$$=\frac{\sin A+\sin B}{\cos B+\cos A}\quad(23)$$
$$=\tan\left(\frac{A+B}{2}\right)\quad(45)$$
$$=\cot\frac{C}{2}\quad(5);$$

$$\therefore\ \sqrt{(1-\cos C)}\,.\,\sqrt{(1+\cos C)}=\frac{\sqrt{(1+\cos C)}}{\sqrt{(1-\cos C)}}\quad(31);$$
$$\therefore\ 1-\cos C=1,\ \text{and}\ \cos C=0;$$
$$\therefore\ C=\frac{\pi}{2}\quad(11).$$

82. If $\text{cosec}\,2B=\dfrac{8\cos^2 A-4}{\text{chd}\,8A}$, the triangle is *either isosceles* or *right-angled.*

Here, $8\cos^2 A-4=\text{chd}\,8A\,.\,\text{cosec}\,2B$,

or $4\,(2\cos^2 A-1)=2\sin 4A\,.\,\text{cosec}\,2B\quad(25)$;

$$\therefore\ 2\cos 2A=\sin 4A\,.\,\text{cosec}\,2B\quad(32)$$
$$=2\sin 2A\,.\,\cos 2A\,.\,\text{cosec}\,2B\quad(29);$$

$$\therefore\ \sin 2A=\frac{1}{\text{cosec}\,2B}=\sin 2B=\sin(\pi-2B);$$

and when $\sin 2A=\sin 2B$

$2A=2B$ and $A=B$,

or the triangle is isosceles. (EUCLID, Prop. 6. Book I.)

And again, when $\sin 2A = \sin(\pi - 2B)$

$$2A = \pi - 2B$$

$$A = \frac{\pi}{2} - B;$$

$$\therefore\ A + B = \frac{\pi}{2};$$

$$\therefore\ C = \pi - (A + B) = \frac{\pi}{2},$$

or the triangle is *right-angled.*

83. If $1 + \cot\left(\frac{\pi}{4} - B\right) = \frac{2}{1 - \tan A}$, and $4K = c^2$, the triangle is *both* isosceles and right-angled.

Here, $1 - \tan A = \dfrac{2}{1 + \cot\left(\frac{\pi}{4} - B\right)}$

$$= \frac{2}{1 + \dfrac{\cot B + 1}{\cot B - 1}} \quad (38) \text{ and } (12)$$

$$= \frac{2 \,.\, (\cot B - 1)}{2 \cot B}$$

$$= 1 - \frac{1}{\cot B}$$

$$= 1 - \tan B;$$

$$\therefore\ \tan A = \tan B, \text{ and } A = B;$$

$\therefore$ also $a = b$ (EUCLID, Prop. 6. Book I.) or the triangle is *isosceles.*

Again,

$$\frac{c^2}{4} = K = \sqrt{\left(\frac{a+b+c}{2}\right).\left(\frac{a+b+c}{2} - a\right).\left(\frac{a+b+c}{2} - b\right).\left(\frac{a+b+o}{2} - c\right)} \quad (49)$$

$$= \sqrt{\left(\frac{2a+c}{2}\right)\frac{c}{2} \cdot \frac{c}{2} \cdot \left(\frac{2a-c}{2}\right)}$$

$$= \frac{c}{4}\sqrt{4a^2 - c^2};$$

$$\therefore\ c=\sqrt{4a^2-c^2}\,; \qquad \therefore\ c^2=4a^2-c^2\,; \qquad \therefore\ 2c^2=4a^2\,;$$

$$\therefore\ C^2=2a^2=a^2+b^2\,;$$

∴ C is a right angle, (EUCLID, Prop. 48. Book I.) or the triangle is *right-angled.*

84. In every right-angled triangle

$$\frac{2\text{ vers }(A-B)}{\text{vers }2A}=\left(\frac{a-b}{a}\right)^2$$

$$\frac{2\text{ vers }(A-B)}{\text{vers }2A}=\frac{2\{1-\cos(A-B)\}}{1-\cos 2A}$$

$$=\frac{1-(\cos A\,.\,\cos B+\sin A\,.\,\sin B)}{\sin^2 A} \quad (32)\text{ and }(37)$$

$$=\frac{1-2\sin A\,.\,\sin B}{\sin^2 A}$$

$$=\frac{1-\dfrac{2ab}{c^2}\,.\,\sin^2 C}{\dfrac{a^2}{c^2}\,.\,\sin^2 C} \quad (48)$$

$$=\frac{1-\dfrac{2ab}{c^2}}{\dfrac{a^2}{c^2}} \quad (12)$$

$$=\frac{c^2-2ab}{a^2}$$

$$=\frac{a^2-2ab+b^2}{a^2} \quad (\text{EUCLID, Prop. 47. Book I.})$$

$$=\left(\frac{a-b}{a}\right)^2.$$

85. In every right-angled triangle,

$$2\text{ cosec }2A\,.\,\cot B=\frac{c^2}{b^2}.$$

$$2\text{ cosec }2A\,.\,\cot B=(\cot A+\tan A)\,.\,\cot B \quad (27)$$

$$=(\tan B+\cot B)\,.\,\cot B \quad (5)\text{ and }(6)$$

$$= \tan B \,.\, \cot B + \cot^2 B$$

$$= 1 + \cot^2 B$$

$$= \operatorname{cosec}^2 B$$

$$= \frac{1}{\sin^2 B}$$

$$= \frac{\sin^2 C}{\sin^2 B} \quad (12)$$

$$= \frac{c^2}{b^2} \quad (48).$$

86. In every right-angled triangle,

$$\tan 2A - \sec 2B = \frac{b+a}{b-a}.$$

$$\tan 2A - \sec 2B = \tan 2A + \sec 2A \quad (7)$$

$$= \frac{1 + \sin 2A}{\cos 2A}$$

$$= \frac{(\cos A + \sin A)^2}{\cos^2 A - \sin^2 A} \quad (23), (29) \text{ and } (30)$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{\sin B + \sin A}{\sin B - \sin A}$$

$$= \frac{\frac{b}{c} + \frac{a}{c}}{\frac{b}{c} - \frac{a}{c}} \quad (48) \text{ and } (12)$$

$$= \frac{b+a}{b-a}.$$

87. In every right-angled triangle,

$$\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}} = \frac{(b+a)c - (b^2+a^2)}{(b-a)c - (b^2-a^2)},$$

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \quad (31)$$

$$= \frac{(1-\cos A)^2}{\sin^2 A};$$

$$\therefore \tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$$

$$= \frac{1-\sin B}{\sin A}$$

$$= \frac{\sin C}{\sin A} - \frac{\sin B}{\sin A} \quad (12)$$

$$= \frac{c-b}{a} \quad (48).$$

Similarly $\tan\frac{B}{2} = \frac{c-a}{b}$;

$$\therefore \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}} = \frac{\frac{c-b}{a} + \frac{c-a}{b}}{\frac{c-b}{a} - \frac{c-a}{b}}$$

$$= \frac{bc - b^2 + ac - a^2}{bc - b^2 - ac + a^2}$$

$$= \frac{(b+a)c - (b^2 + a^2)}{(b-a)c - (b^2 - a^2)}.$$

88. The cotangents of the semi-angles of a plane triangle are consecutive natural numbers. Required the nature of the triangle and the proportion of its sides.

Let $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ be respectively denoted by $x+1$, x, $x-1$.

$$\text{Now } A + B + C = \pi; \quad \therefore \frac{A}{2} + \frac{C}{2} = \frac{\pi}{2} - \frac{B}{2};$$

$$\therefore \cot\left(\frac{A}{2} + \frac{C}{2}\right) = \tan\frac{B}{2},$$

$$\text{or } \frac{(x+1)\,.\,(x-1) - 1}{(x+1) + (x-1)} = \frac{1}{x} \quad (38),$$

$$\text{or } \frac{x^2-2}{2x}=\frac{1}{x}, \text{ or } x^2-2=2; \quad \therefore x^2=4 \text{ and } x=\pm 2,$$

where the *negative* sign must be rejected, because in that case the cotangents of the semi-angles being negative, the semi-angles themselves must each be greater than a right angle, which is impossible;

$$\therefore x=2 \text{ and } \cot\frac{C}{2}=x-1=1=\cot\frac{\pi}{4} \quad (12);$$

$$\therefore \frac{C}{2}=\frac{\pi}{4} \text{ and } C=\frac{\pi}{2}; \quad \therefore \text{the triangle is right-angled.}$$

$$\text{Again, } 2=\cot\frac{B}{2}\,.\,\cot\frac{C}{2}$$

$$=\sqrt{\frac{S\,.\,(S-b)}{(S-a)\,(S-c)}}\,.\,\sqrt{\frac{S\,.\,(S-c)}{(S-a)\,.\,(S-b)}} \quad (53)$$

$$=\frac{S}{S-a};$$

$$\therefore 2S-2a=S, \text{ and } S=2a.$$

$$\text{Also } 3=\cot\frac{A}{2}\,.\,\cot\frac{C}{2}$$

$$=\sqrt{\frac{S\,.\,(S-a)}{(S-b)\,.\,(S-c)}}\,.\,\sqrt{\frac{S\,.\,(S-c)}{(S-a)\,.\,(S-b)}} \quad (53)$$

$$=\frac{S}{S-b}=\frac{2a}{2a-b};$$

$$\therefore 6a-3b=2a;$$

$$\therefore 4a=3b, \text{ and } b=\frac{4}{3}a.$$

But $c^2=a^2+b^2$ (EUCLID, Prop. 47. Book I.) $=a^2+\frac{16}{9}a^2=\frac{25}{9}a^2$;

$$\therefore c=\frac{5}{3}a,$$

$$\text{and } a=\frac{3}{3}a;$$

$$\therefore a,\ b,\ c \text{ are as } 3,\ 4,\ 5,$$

and the triangle is a right-angled one whose sides are as the numbers 3, 4, 5.

89. $$\tan B + \tan A = 2 \operatorname{cosec} \frac{A+B}{2},$$

$$\text{and } \tan^2 B - \tan^2 A = \frac{8 \cos\left(\frac{A+B}{2}\right)}{1 - \cos(A+B)},$$

required the nature of the triangle and the proportion of its angles.

$$\tan B - \tan A = \frac{\tan^2 B - \tan^2 A}{\tan B + \tan A}$$

$$= \frac{8 . \cos\left(\frac{A+B}{2}\right) . \sin\left(\frac{A+B}{2}\right)}{4 \sin^2\left(\frac{A+B}{2}\right)} \quad (23) \text{ and } (32)$$

$$= 2 \cot\left(\frac{A+B}{2}\right)$$

$$= 2 \tan \frac{C}{2} \quad (5)$$

$$\text{and } \tan B + \tan A = 2 \sec \frac{C}{2} \quad (7);$$

$$\therefore \tan B = \sec \frac{C}{2} + \tan \frac{C}{2},$$

$$\tan A = \sec \frac{C}{2} - \tan \frac{C}{2};$$

$$\therefore \tan A . \tan B = \sec^2 \frac{C}{2} - \tan^2 \frac{C}{2} = 1 \quad (23) \quad = \tan B . \cot B \quad (23);$$

$$\therefore \tan A = \cot B = \tan\left(\frac{\pi}{2} - B\right);$$

$$\therefore A = \frac{\pi}{2} - B, \text{ and } A + B = \frac{\pi}{2};$$

$$\therefore C = \pi - (A+B) = \frac{\pi}{2} \text{ and the triangle is right-angled,}$$

$$\text{and } \tan A = \sec \frac{\pi}{4} - \tan \frac{\pi}{4} = \sqrt{2} - 1 \quad (16) \text{ and } (12)$$

$$\tan B = \sec \frac{\pi}{4} + \tan \frac{\pi}{4} = \sqrt{2} + 1;$$

$$\therefore\ A = 22^\circ.30' \quad (15) \quad = 1\,(22^\circ.30')$$
$$B = 67^\circ.30' \quad (15) \quad = 3\,(22^\circ.30')$$
$$C = 90 \quad = 4\,(22^\circ.30').$$

Therefore the triangle is a right-angled one, having its angles as the numbers 1, 3, 4.

90. The area of a triangle whose sides are $2n^2+2$, $2n^2$ and $2n^2-2$ is four times that of the triangle whose sides are $2n+1$, $2n$ and $2n-1$, find all the values of (n), and solve the greater triangle when $n=\pm 2$.

The area of a Δ (whose sides are $2n^2+2$, $2n^2$, $2n^2-2$)

$$=\sqrt{3n^2.(n^2-2)n^2.(n^2+2)} \quad (49)$$

.... $2n+1$, $2n$, $2n-1$

$$=\sqrt{3n\,.\,(n-1)\,n\,.\,(n+1)}\,;$$
$$\therefore\ \sqrt{3n^2.(n^2-2)\,n^2.(n^2+2)} = 4\,.\sqrt{3n\,.(n-1)\,n\,.(n+1)},$$
$$\text{or } n^6-4n^2=16n^2-16\,;$$
$$\therefore\ n^6-16n^2-(4n^2-16)=0\,;$$
$$\therefore\ (n^2-4)(n^4+4n^2)-4\,.\,(n^2-4)=0,$$
$$\text{or } (n^4+4n^2-4)(n^2-4)=0\,;\quad \therefore\ n^4+4n^2-4=0, \text{ and } n^2-4=0.$$

Taking the biquadratic factor $n^4+4n^2-4=0$, we have

$$n^4+4n^2=4\,;\quad \therefore\ n^4+4n^2+4=8\,;$$
$$\therefore\ n^2+2=\pm 2\sqrt{2}\,;$$
$$\therefore\ n^2=-2\pm 2\sqrt{2}\,;\quad n=\pm\sqrt{(-2\pm 2\sqrt{2})}.$$

Again, $n^2-4=0$; $\therefore\ n^2=4$; $\therefore\ n=\pm 2$.

And when $n=\pm 2$, the *sides* of the greater triangle are 10, 8 and 6.

Let C, B, A be the angles respectively opposite these sides.

$$\text{Then } \cos C=\frac{64+36-100}{2\,.\,8\,.\,6}=0 \quad (50)\,;$$
$$\therefore\ C=90 \quad (11)\,;\quad \therefore\ A+B=90.$$
$$\text{And } \tan A=\frac{\sin A}{\cos A}=\frac{\sin A}{\sin B}=\frac{8}{6} \quad (48) \quad =\frac{4}{3} \text{ to rad. } 1,$$

and $\log \tan A = 10 + \log_{10} 4 - \log_{10} 3$ to tabular radius

$$= 10 + .60206 - .47712$$

$$= 10\,.\,12494 = \log \tan 53°.\,7'.\,54'' \text{ (by the tables)};$$

$$\therefore\ A = 53°.\ 7'.\,54'',$$

$$\text{and } B = 36°.\,52'.\ 6'',$$

$$C = 90.$$

91. $R_1 R_2 = c$, and $R_2 = a - b$. Solve the triangle.

Let a, b be denoted by $x + y$, x.

Then c will .... $\sqrt{(2x^2 + 2xy + y^2)}$;

and $\therefore$, $R_2 = x + y - x = y$, and $R_1 R_2 = \sqrt{(2x^2 + 2xy + y^2)}$.

$$\text{But } 2R_1 = \sqrt{(2x^2 + 2xy + y^2)}\ (46) = R_1 R_2;$$

$$\therefore\ 2 = R_2 = y.$$

Again $2R_2 = (x + y) + x - \sqrt{(2x^2 + 2xy + y^2)}$. (BLAND's *Deductions*, Section VII. Prop. 3.)

$$\text{or } 4 = 2x + 2 - \sqrt{(2x^2 + 4x + 4)};$$

$$\therefore\ \sqrt{(2x^2 + 4x + 4)} = 2x - 2;$$

$$\therefore\ 2x^2 + 4x + 4 = 4x^2 - 8x + 4;$$

$$\therefore\ 2x^2 = 12x, \text{ and } x = 6;$$

$\therefore$ the sides are 10, 8 and 6,

and the angles 90°; 53°. 7'. 54"; 36°. 52'. 6". (Prop. 90.)

92. The difference between the first and second, and also between the second and third sides of a right-angled triangle is a given square number (n^2). Find the sides.

Let $x + n^2$ be the hypothenuse, x, and $x - n^2$ the sides.

Then $(x + n^2)^2 = x^2 + (x - n^2)^2$. (EUCLID, Prop. 47. Book I.);

$$\therefore\ x^2 = (x + n^2)^2 - (x - n^2)^2 = 4n^2 x;$$

$$\therefore\ x = 4n^2;$$

$\therefore$ the sides of the triangle are $5n^2$, $4n^2$, and $3n^2$.

COR. If as a particular case of this Problem, we have $4S = K$, then $n = \pm 2$.

$$\text{For } 5n^2 + 4n^2 + 3n^2 = 2S = \frac{K}{2} = \frac{4n^2 . 3n^2}{4} . \sin C \quad (49),$$

$$\text{or } 12n^2 = 3n^4 \quad (12);$$

$$\therefore\ n^2 = 4, \quad \text{and } n = \pm 2.$$

93. Find right-angled triangles such that the difference between the hypothenuse and the least side may be a square number.

Let $x + 2y^2$, x, $x - 2y^2$ be the sides.

Then $x + 2y^2 - (x - 2y^2) = 4y^2$ a square number.

Also $(x + 2y^2)^2 = x^2 + (x - 2y^2)^2$. (Euclid, Prop. 47. Book I.)

$$\therefore\ x^2 = (x + 2y^2)^2 - (x - 2y^2)^2 = 8xy^2;$$

$$\therefore\ x = 8y^2,$$

and the sides are $10y^2$, $8y^2$, $6y^2$, where y may be assumed at pleasure.

Let $y = 1$. Then sides are 10, 8, 6.
$y = 2$. .... 40, 32, 24.
$y = 3$. .... 90, 72, 54.
&c. = &c. ....

CHAPTER IV.

OBLIQUE-ANGLED TRIANGLES.

94. $$\frac{(\text{Cot}\,A + \cot C)\,.\,(\cot B + \cot C)}{\operatorname{cosec}^2 C} = 1,$$

$$\frac{(\cot A + \cot C)\,.\,(\cot B + \cot C)}{\operatorname{cosec}^2 C}$$

$$= \frac{\cot A\,.\,\cot B + \cot A\,.\,\cot C + \cot B\,.\,\cot C + \cot^2 C}{\operatorname{cosec}^2 C}$$

$$= \frac{1 + \cot^2 C}{\operatorname{cosec}^2 C} \quad (47)$$

$$= \frac{\operatorname{cosec}^2 C}{\operatorname{cosec}^2 C} \quad (23)$$

$$= 1.$$

95. $$\frac{\cos^2\left(\frac{A+B}{4}\right) - \sin^2\left(\frac{A+B}{4}\right)}{\cot^2\left(\frac{A+B}{4}\right) - \tan^2\left(\frac{A+B}{4}\right)} = \frac{1}{4}\,.\,\cos^2\frac{C}{2}.$$

Let $\frac{A+B}{4} = \theta$. Then $\frac{A+B}{2} = 2\theta$.

Now $\cos^2\theta - \sin^2\theta = \cos 2\theta \quad (30)$,

and $\cot^2\theta - \tan^2\theta = (\cot\theta + \tan\theta)\,.\,(\cot\theta - \tan\theta)$

$$= 2\operatorname{cosec} 2\theta\,.\,2\cot 2\theta \quad (27)$$

$$= \frac{4\cos 2\theta}{\sin^2 2\theta};$$

$$\therefore \frac{\cos^2\theta - \sin^2\theta}{\cot^2\theta - \tan^2\theta} = \frac{1}{4}\,.\,\sin^2 2\theta,$$

and by substitution $\dfrac{\cos^2\left(\dfrac{A+B}{4}\right)-\sin^2\left(\dfrac{A+B}{4}\right)}{\cot^2\left(\dfrac{A+B}{4}\right)-\tan^2\left(\dfrac{A+B}{4}\right)}=\dfrac{1}{4}\sin^2\left(\dfrac{A+B}{2}\right)$

$$=\frac{1}{4}\cdot\cos^2\frac{C}{2}\quad(1).$$

96. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=\text{chd}\,A\,.\,\text{chd}\,B\,.\,\text{chd}\,C,$

$$\sin 2A+\sin 2B+\sin 2C$$

$= 2\sin(A+B)\,.\,\cos(A-B)+2\sin C\,.\,\cos C$ (40) and (29)

$= 2\sin C\,.\,\{\cos(A-B)-\cos(A+B)\}$ (3) and (4)

$= 2\sin C\,(2\sin A\,.\,\sin B)$ (42)

$= 4\sin A\,.\,\sin B\,.\,\sin C.$

Again, $\sin A+\sin B+\sin C$

$=2\sin\left(\dfrac{A+B}{2}\right)\,.\,\cos\left(\dfrac{A-B}{2}\right)+2\sin\dfrac{C}{2}\,.\,\cos\dfrac{C}{2}$ (40) and (29)

$=2\cos\dfrac{C}{2}\left\{\cos\left(\dfrac{A-B}{2}\right)+\cos\left(\dfrac{A+B}{2}\right)\right\}$ (1)

$=4\cos\dfrac{C}{2}\,.\,\cos\dfrac{A}{2}\,.\,\cos\dfrac{B}{2}$ (42);

$$\therefore\ \frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$$

$$=\frac{\sin A\,.\,\sin B\,.\,\sin C}{\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\,\cos\frac{C}{2}}$$

$$=\frac{2\sin\frac{A}{2}\,.\,\cos\frac{A}{2}\,.\,2\sin\frac{B}{2}\,.\,\cos\frac{B}{2}\,.\,2\sin\frac{C}{2}\,.\,\cos\frac{C}{2}}{\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\,\cos\frac{C}{2}}\quad(29)$$

$$=2\sin\frac{A}{2}\,.\,2\sin\frac{B}{2}\,.\,2\sin\frac{C}{2}$$

$= \text{chd}\,A\,.\,\text{chd}\,B\,.\,\text{chd}\,C$ (25).

97. $$\frac{\sin 2A + \sin 2B + \sin 2C}{\tan A + \tan B + \tan C} = 4\cos A \cdot \cos B \cdot \cos C,$$

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\tan A + \tan B + \tan C}$$

$$= \frac{4\sin A \cdot \sin B \cdot \sin C}{\tan A \cdot \tan B \cdot \tan C} \quad (47) \text{ and (Prop. 96)}$$

$$= 4\cos A \cdot \cos B \cdot \cos C \quad (26).$$

98. From the expression for the area of a triangle in terms of the sides deduce the following.

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}.$$

Since $K^2 = S \cdot (S-a)(S-b)(S-c)$

$$= (S-a)(S-b)(S-c) \cdot (3S - 2S)$$

$$= (S-a)(S-b)(S-c) \cdot \{(S-a) + (S-b) + (S-c)\}$$

$$= (S-a)^2 \cdot (S-b)(S-c) + (S-b)^2 \cdot (S-a)(S-c) + (S-c)^2 \cdot (S-a) \cdot (S-b);$$

$$\therefore \frac{K^2}{(S-a)^2 \cdot (S-b)^2 \cdot (S-c)^2} = \frac{1}{(S-b) \cdot (S-c)} + \frac{1}{(S-a)(S-c)} + \frac{1}{(S-a)(S-b)};$$

$$\therefore \frac{K^3}{(S-a)^2 \cdot (S-b)^2 \cdot (S-c)^2} = \frac{K}{(S-b) \cdot (S-c)} + \frac{K}{(S-a) \cdot (S-c)} + \frac{K}{(S-a) \cdot (S-b)}.$$

But $K = \frac{bc}{2} \cdot \sin A = \frac{ac}{2} \cdot \sin B = \frac{ab}{2} \cdot \sin C$ (49),

$$\text{and } (S-a)(S-b) = ab \cdot \sin^2\frac{C}{2},$$

$$(S-a)(S-c) = ac \,.\, \sin^2 \frac{B}{2} \quad (51),$$

$$(S-b)(S-c) = bc \,.\, \sin^2 \frac{A}{2}.$$

Hence we obtain, by the substitution of these values,

$$\frac{\sin A}{2\sin^2 \frac{A}{2}} + \frac{\sin B}{2\sin^2 \frac{B}{2}} + \frac{\sin C}{2\sin^2 \frac{C}{2}} = \frac{\sin A}{2\sin^2 \frac{A}{2}} \cdot \frac{\sin B}{2\sin^2 \frac{B}{2}} \cdot \frac{\sin C}{2\sin^2 \frac{C}{2}};$$

$$\therefore \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} + \frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} = \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} \cdot \frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} \cdot \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \quad (29);$$

$$\text{or } \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \,.\, \cot \frac{B}{2} \,.\, \cot \frac{C}{2} \quad (26)$$

99. From the Formula

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \,.\, \cos \frac{B}{2} \,.\, \cos \frac{C}{2},$$

deduce the expression for the area of a triangle in terms of the sides.

$$\text{Since } \sin A = \frac{2K}{bc}, \quad \sin B = \frac{2K}{ac}, \quad \sin C = \frac{2K}{ab} \quad (49);$$

$$\text{and } \cos \frac{A}{2} = \sqrt{\frac{S\,.\,(S-a)}{bc}}, \quad \cos \frac{B}{2} = \sqrt{\frac{S\,.\,(S-b)}{ac}},$$

$$\cos \frac{C}{2} = \sqrt{\frac{S\,.\,(S-c)}{ab}} \quad (52);$$

$$\therefore \frac{2K}{bc} + \frac{2K}{ac} + \frac{2K}{ab} = 4 \,.\, \sqrt{\left\{\frac{S^3\,.\,(S-a)\ldots(S-b)\ldots(S-c)}{a^2b^2c^2}\right\}},$$

by substitution;

$$\text{or } K\left(\frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}\right) = \frac{2S}{abc} \cdot \sqrt{S\,.\,(S-a)(S-b)(S-c)};$$

$$\therefore\ K(a+b+c) = 2S\,.\,\sqrt{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)}.$$

$$\text{But}\ \ 2S = a + b + c\,;$$

$$\therefore\ K = \sqrt{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)}.$$

100.
$$\frac{\cot A + \cot B + \cot C - 2\,(\cot 2A + \cot 2B + \cot 2C)}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}$$

$$= (\sec A - 1)\,.\,(\sec B - 1)\,.\,(\sec C - 1),$$

$$\frac{\cot A + \cot B + \cot C - 2\,(\cot 2A + \cot 2B + \cot 2C)}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}$$

$$= \frac{(\cot A - 2\cot 2A) + (\cot B - 2\cot 2B) + (\cot C - 2\cot 2C)}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}$$

$$= \frac{\tan A + \tan B + \tan C}{\cot \frac{A}{2}\,.\,\cot \frac{B}{2}\,.\,\cot \frac{C}{2}} \quad (27)\ \text{and (Prop. 98.)}$$

$$= \frac{\tan A\,.\,\tan B\,.\,\tan C}{\cot \frac{A}{2}\,.\,\cot \frac{B}{2}\,.\,\cot \frac{C}{2}} \quad (47)$$

$$= \frac{\dfrac{\sin A}{\cos A}\,.\,\dfrac{\sin B}{\cos B}\,.\,\dfrac{\sin C}{\cos C}}{\dfrac{\cos \frac{A}{2}}{\sin \frac{A}{2}}\,.\,\dfrac{\cos \frac{B}{2}}{\sin \frac{B}{2}}\,.\,\dfrac{\cos \frac{C}{2}}{\sin \frac{C}{2}}}$$

$$= \frac{2\sin^2 \frac{A}{2}\,.\,2\sin^2 \frac{B}{2}\,.\,2\sin^2 \frac{C}{2}}{\cos A\,.\,\cos B\,.\,\cos C} \quad (29)$$

$$= \frac{(1-\cos A)\,.\,(1-\cos B)\,.\,(1-\cos C)}{\cos A\,.\,\cos B\,.\,\cos C} \quad (32)$$

$$= (\sec A \quad 1)\,.\,(\sec B - 1)\,.\,(\sec C - 1) \quad (23).$$

101. $$\frac{S^3}{abc} = \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\sec\frac{A}{2} \cdot \sec\frac{B}{2} \cdot \sec\frac{C}{2}},$$

$$\frac{a+b+c}{a} = 1 + \frac{b}{a} + \frac{c}{a}$$

$$= \frac{\sin A}{\sin A} + \frac{\sin B}{\sin A} + \frac{\sin C}{\sin A} \quad (48)$$

$$= \frac{\sin A + \sin B + \sin C}{\sin A}$$

$$= \frac{4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}}{2\sin\frac{A}{2} \cdot \cos\frac{A}{2}} \quad (29) \text{ and (Prop. 96.)}$$

$$= \frac{2\cos\frac{B}{2} \cdot \cos\frac{C}{2}}{\sin\frac{A}{2}}.$$

Similarly $$\frac{a+b+c}{b} = \frac{2\cos\frac{A}{2} \cdot \cos\frac{C}{2}}{\sin\frac{B}{2}},$$

and $$\frac{a+b+c}{c} = \frac{2\cos\frac{A}{2} \cdot \cos\frac{B}{2}}{\sin\frac{C}{2}};$$

$$\therefore \frac{(a+b+c)^3}{abc} = \frac{2^3 \cdot \cos^2\frac{A}{2} \cdot \cos^2\frac{B}{2} \cdot \cos^2\frac{C}{2}}{\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}};$$

$$\therefore \frac{S^2}{abc} = \frac{\cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}}{\sec\frac{A}{2} \cdot \sec\frac{B}{2} \cdot \sec\frac{C}{2}} \quad (23) \text{ and } (26)$$

$$= \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\sec\frac{A}{2} \cdot \sec\frac{B}{2} \cdot \sec\frac{C}{2}} \quad \text{(Prop. 98.)}$$

102. $$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{S^2}{K},$$

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\,.\,\cot\frac{B}{2}\,.\,\cot\frac{C}{2}\quad\text{(Prop. 98.)}$$

$$=\sqrt{\frac{S\,.\,(S-a)}{(S-b)(S-c)}}\,.\,\sqrt{\frac{S\,.\,(S-b)}{(S-a)(S-c)}}\,.\,\sqrt{\frac{S\,.\,(S-c)}{(S-a)(S-b)}}\quad(53)$$

$$=\frac{S\,.\,\sqrt{S}}{\sqrt{\{(S-a)\,.\,(S-b)\,.\,(S-c)\}}}$$

$$=\frac{S^2}{\sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}}$$

$$=\frac{S^2}{K}\quad(49).$$

103. $$\frac{\cos B-\cos A}{1+\cos C}=\frac{a-b}{c}$$

$$\frac{\cos B-\cos A}{1+\cos C}=\frac{2\sin\left(\frac{A+B}{2}\right)\,.\,\sin\left(\frac{A-B}{2}\right)}{2\cos^2\frac{C}{2}}\quad(32)\text{ and }(40)$$

$$=\frac{2\sin\left(\frac{A+B}{2}\right)\,.\,\sin\left(\frac{A-B}{2}\right)}{2\sin^2\left(\frac{A+B}{2}\right)}\quad(2)$$

$$=\frac{2\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)}$$

$$=\frac{2\cos\left(\frac{A+B}{2}\right)\,.\,\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\,.\,\cos\left(\frac{A+B}{2}\right)}$$

$$=\frac{\sin A-\sin B}{\sin(A+B)}\quad(29)\text{ and }(40)$$

$$=\frac{\sin A-\sin B}{\sin C}\quad(3)$$

$$=\frac{a-b}{c}\quad(48).$$

104. $\dfrac{\cot B - \cot A}{\operatorname{cosec} C} = \tan\theta \,.\, \sin\theta$, where $\sec\theta = \dfrac{a}{b}$,

$$\frac{\cot B - \cot A}{\operatorname{cosec} C} = \left(\frac{\cos B}{\sin B} - \frac{\cos A}{\sin A}\right).\sin C \quad (23) \text{ and } (26)$$

$$= \cos B \,.\, \frac{\sin C}{\sin B} - \cos A \,.\, \frac{\sin C}{\sin A}$$

$$= \frac{a^2 + c^2 - b^2}{2ac}\,.\,\frac{c}{b} - \frac{b^2 + c^2 - a^2}{2bc}\,.\,\frac{c}{a} \quad (48) \text{ and } (50)$$

$$= \frac{a^2 + c^2 - b^2 - b^2 - c^2 + a^2}{2ab}$$

$$= \frac{a^2 - b^2}{ab}$$

$$= \frac{a}{b} - \frac{b}{a}$$

$$= \sec\theta - \frac{1}{\sec\theta} \quad \text{(by hypothesis)}$$

$$= \frac{\sec^2\theta - 1}{\sec\theta}$$

$$= \frac{\tan^2\theta}{\sec\theta}$$

$$= \tan\theta \,.\, \frac{\tan\theta}{\sec\theta}$$

$$= \tan\theta \,.\, \sin\theta \quad (25).$$

105. If $\operatorname{chd} C = \dfrac{c}{a}$, the triangle is *isosceles*.

Since $\operatorname{chd} C = \dfrac{c}{a}$;

$$\therefore\ 2\sin\frac{C}{2} = \frac{c}{a}$$

$$= \frac{\sin C}{\sin A} \quad (48)$$

$$= \frac{2\sin\frac{C}{2}\,.\,\cos\frac{C}{2}}{\sin A} \quad (29);$$

$$\therefore \ \sin A = \cos\frac{C}{2}$$

$$= \sin\left(\frac{A+B}{2}\right);$$

$$\therefore \ A = \frac{A+B}{2};$$

$$\therefore \ 2A = A + B;$$

$$\therefore \ A = B;$$

$\therefore \ a = b$; (EUCLID, Prop. 6, Book I.) and the triangle is isosceles.

106. If $\operatorname{cosec} B + \cot B = \frac{2a+c}{\sqrt{(4a^2-c^2)}}$, the triangle is *isosceles*.

$\operatorname{cosec} B + \cot B = \cot\frac{B}{2}$ (28), and $\frac{2a+c}{\sqrt{4a^2-c^2}} = \sqrt{\frac{2a+c}{2a-c}}$;

$$\therefore \ \frac{2a+c}{2a-c} = \cot^2\frac{B}{2} = \frac{1+\cos B}{1-\cos B} \quad (31);$$

$$\therefore \ 2a + c - (2a+c)\,.\cos B = 2a - c + (2a-c)\,.\cos B;$$

$$\therefore \ 2a + c - (2a-c) = \{(2a+c) + (2a-c)\}\,.\cos B,$$

$$\text{or } 2c = 4a\,.\cos B = 4a\,.\frac{a^2+c^2-b^2}{2ac} \quad (50);$$

$\therefore \ c^2 = a^2 + c^2 - b^2$; $\therefore \ a^2 = b^2$, and $a = b$, or the triangle is isosceles.

107. If $\frac{a^3+b^3+c^3}{a+b+c} = c^2$, and $\sin A\,.\sin B = \sin^2 C$, the triangle is *equilateral*.

$$a^3 + b^3 + c^3 = (a+b)c^2 + c^3;$$

$$\therefore \ c^2 = \frac{a^3+b^3}{a+b}$$

$$= a^2 - ab + b^2.$$

$$\text{Also } \frac{c^2}{ab} = \frac{\sin^2 C}{\sin A\,.\sin B} \quad (48)$$

$$= 1, \text{ by hypothesis};$$

$$\therefore\ ab = c^2 = a^2 - ab + b^2;$$

$$\therefore\ a^2 - 2ab + b^2 = 0;$$

$$\therefore\ a - b = 0;$$

$$\therefore\ a = b;$$

$$\therefore\ c^2 = ab = a^2 = b^2,$$

and $c = a = b$;

$\therefore$ the triangle is equilateral.

108. Given $\sec 2B + \tan 2A = \tan\left(\frac{\pi}{4} + A\right)$, and $1 - \cos 2C$

$$= \frac{\cos 2C - \cos 2A}{(2\cos C + 1)\,.\,(2\cos C - 1)}.$$

Required the nature of the triangle and the proportion of its angles.

$$\text{Since }\ 1 - \cos 2C = \frac{\cos 2C - \cos 2A}{(2\cos C + 1)\,.\,(2\cos C - 1)};$$

$$\therefore\ 4\cos^2 C - 1 = \frac{\cos 2C - \cos 2A}{1 - \cos 2C}$$

$$= \frac{1 - 2\sin^2 C - (1 - 2\sin^2 A)}{2\sin^2 C} \qquad (32)$$

$$= \frac{\sin^2 A - \sin^2 C}{\sin^2 C}$$

$$= \frac{\sin^2 A}{\sin^2 C} - 1;$$

$$\therefore\ 4\cos^2 C = \frac{\sin^2 A}{\sin^2 C},$$

and $\sin A = 2\sin C\,.\,\cos C = \sin 2C$;

$$\therefore\ A = 2C.$$

$$\text{Again }\ \sec 2B = \tan\left(\frac{\pi}{4} + A\right) - \tan 2A$$

$$= \frac{1 + \tan A}{1 - \tan A} - \frac{2\tan A}{1 - \tan^2 A} \qquad (12),\ (33) \text{ and } (38)$$

$$= \frac{1 + \tan^2 A}{1 - \tan^2 A}$$

$$= \sec 2A \qquad (23) \text{ and } (30);$$

$$\therefore\ 2B = 2A, \quad\text{and}\quad B = A = 2C;$$

consequently the triangle is an isosceles triangle (EUCLID, Book IV. Prop. 10.) having each of the angles at the base double that at the vertex.

109. Given $4S = \tan^2\left(\frac{A+B}{2}\right)$; $a - b = 1$ and $b = c^2$;

find the sides.

$$2(a+b+c) = 4S = \tan^2\left(\frac{A+B}{2}\right) = \cot^2\frac{C}{2} \quad (5)$$

$$= \frac{(a+b+c)(a+b-c)}{(a+c-b)(b+c-a)} \quad (53);$$

$$\therefore\ 2 = \frac{a+b-c}{\{c+(a-b)\}\{c-(a-b)\}} = \frac{2b+1-c}{c^2-(a-b)^2} = \frac{2c^2+1-c}{c^2-1},$$

by hypothesis;

$$\therefore\ 2c^2 - 2 = 2c^2 + 1 - c;$$

$$\therefore\ c = 3,$$

$$b = 9,$$

$$a = 10.$$

110. The sides of certain triangles consist of two digits each. The third side is the first with inverted digits, and the second is 10 times the right hand digit of the first. Also

$$\cos\frac{A}{2} = 12 \,.\, \sin\left(\frac{C-B}{2}\right),$$

where A, B, C are the angles opposite the first, second, and third sides respectively. Find the sides of all the triangles which answer the conditions of the question.

Let x be the left and y the right hand digit of the first side. Then $10x + y$, $10y$, $10y + x$ will be the sides.

$$\text{Now } \frac{1}{12} = \frac{\sin\left(\frac{C-B}{2}\right)}{\cos\frac{A}{2}}$$

$$= \frac{2 \sin\left(\frac{C-B}{2}\right) . \cos\left(\frac{C+B}{2}\right)}{2 \sin \frac{A}{2} . \cos \frac{A}{2}} \quad (2)$$

$$= \frac{\sin C - \sin B}{\sin A} \quad (40) \text{ and } (29)$$

$$= \frac{(10y + x) - 10y}{10x + y} \quad (48)$$

$$= \frac{x}{10x + y};$$

$$\therefore \; 10x + y = 12x;$$

$$\therefore \; y = 2x;$$

therefore the sides are $12x$, $20x$, $21x$ where, in order that the sides may consist of *two* digits each, (x) must be taken < 5.

Let $x = 1$. Then sides are 12, 20, 21,
$x = 2$ 24, 40, 42,
$x = 3$ 36, 60, 63,
$x = 4$ 48, 80, 84,

the sides of *all* the triangles which answer the conditions of the question.

111. The first, third, sum of the first and second, and difference of the third and second sides of a plane triangle are all *integral* and *square* numbers. Find them, and shew that

$$\frac{\sin(B + C)}{\sin(A + B) - \sin(A + C)} = \left\{2 \cot \frac{C}{2} . \cot\left(\frac{B - A}{2}\right)\right\}^2,$$

where A, B, C are the angles opposite the first, second and third sides respectively.

Let x^2, $3x^2$ and $(2x - 1)^2$ be the sides.

Then x^2, $(2x - 1)^2$ and $x^2 + 3x^2 = 4x^2$ are square numbers.

But $(2x - 1)^2 - 3x^2$ must also be a square number.

Let $\therefore$ $(2x - 1)^2 - 3x^2 = (x - y)^2$.

Then $x^2 - 4x + 1 = x^2 - 2xy + y^2$;

$$\therefore\ 2xy - 4x = y^2 - 1\,;$$

$$\therefore\ x = \frac{y^2 - 1}{2y - 4},$$

where (y) may be any number greater than 2 that renders (x) an integer, which is the case both with 3 and 5.

$$\text{Let } \therefore\ y = 3 \text{ or } 5, \text{ then } x = 4\,;$$

$$\text{and } \therefore \text{ the sides are } 16, 48 \text{ and } 49.$$

$$\text{Again } \frac{\sin(B+C)}{\sin(A+B) - \sin(A+C)} = \frac{\sin A}{\sin C - \sin B} \quad (3)$$

$$= \frac{16}{49 - 48} \quad (48)$$

$$= 16.$$

$$\text{Also } \cot\frac{C}{2}\,.\,\cot\left(\frac{B-A}{2}\right) = \frac{\tan\left(\frac{B+A}{2}\right)}{\tan\left(\frac{B-A}{2}\right)} \quad (6) \text{ and } (23)$$

$$= \frac{48 + 16}{48 - 16} \quad (54)$$

$$= \frac{64}{32}$$

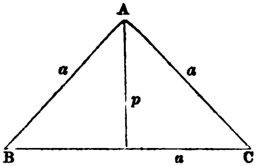
$$= 2\,;$$

$$\therefore\ 2\cot\frac{C}{2}\,.\,\cot\left(\frac{B-A}{2}\right) = 4\,;$$

$$\therefore\ \left\{2\cot\frac{C}{2}\,.\,\cot\left(\frac{B-A}{2}\right)\right\}^2 = 16 = \frac{\sin(B+C)}{\sin(A+B) - \sin(A+C)}.$$

112. A perpendicular (p) is drawn from the vertical angle (A) of an equilateral triangle ABC whose side is (a) and area (K). On this perpendicular a second equilateral triangle is described, and on the perpendicular of this a third and so on *ad infinitum*. Shew the sum of all these triangles to be three times the area of the given triangle.

Let K_1, K_2, K_3, K_4...... be the areas of the triangles on the successive perpendiculars.



Now $A+B+C=180$. But $A=B=C$. (Prop. 5, Cor. EUCLID, Book I.)

$$\therefore\ 3A=180;\quad \therefore\ A=60=B=C;$$

and so of all the angles of the other equilateral triangles;

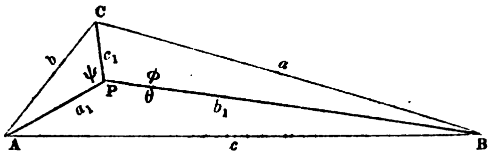
$$\begin{aligned}\therefore\ K : K_1 &:: \frac{a^2}{2}\sin 60 : \frac{p^2}{2}.\sin 60 \quad (49)\\ &:: a^2 : p^2\\ &:: 1 : \sin^2 60\\ &:: 1 : \tfrac{3}{4} \quad (17);\\ \therefore\ K_1 &= \tfrac{3}{4}K.\end{aligned}$$

$$\begin{aligned}\text{Similarly}\ K_2 &= \tfrac{3}{4}K_1 = (\tfrac{3}{4})^2 K,\\ K_3 &= \tfrac{3}{4}K_2 = (\tfrac{3}{4})^3 K,\\ \&c. &= \&c.\end{aligned}$$

$$\begin{aligned}\therefore\ K_1+K_2+K_3+\ldots\ldots &= \{\tfrac{3}{4}+(\tfrac{3}{4})^2+(\tfrac{3}{4})^3+\ldots\ldots\}K\\ &= \left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right)K \quad (\text{HIND's } \textit{Algebra}, 270.)\\ &= 3K.\end{aligned}$$

113. If P be a point within the triangle ABC such that $\dfrac{a^2-b^2}{b_1-a_1}=a_1+b_1+c_1$, and $a^2=b_1^2+b_1c_1+c_1^2$, AP, BP, CP being respectively denoted by a_1, b_1, c_1, then the sides of the triangle will subtend equal angles at P.

Let the angles subtended by AB, BC, AC be θ, ϕ, ψ.



Then $b_1^2 - a_1^2 + b_1c_1 - a_1c_1 = a^2 - b^2$,

and $b_1^2 + c_1^2 + b_1c_1 \qquad = a^2$;

$$\therefore\ a_1^2 + c_1^2 + a_1c_1 = b^2 = a_1^2 + c_1^2 - 2a_1c_1 . \cos\psi \quad (50);$$

$$\therefore\ a_1c_1 = -2a_1c_1 . \cos\psi;$$

$$\therefore\ \cos\psi = -\tfrac{1}{2} = -\cos 60 \ (14) = \cos 120 \quad (4);$$

$$\therefore\ \psi = 120.$$

Again $b_1^2 + c_1^2 + b_1c_1 = a^2 = b_1^2 + c_1^2 - 2b_1c_1 . \cos\phi$;

$$\therefore\ b_1c_1 = -2b_1c_1 . \cos\phi;$$

$$\therefore\ \cos\phi = -\tfrac{1}{2}, \text{ and } \phi = 120.$$

But $\theta + \phi + \psi = 360$. (Prop. 15, Cor. 2, Euclid, Book I.)

$$\therefore\ \theta = 360 - (\phi + \psi) = 360 - 240 = 120;$$

$$\therefore\ \theta = \phi = \psi.$$

CHAPTER V.

EQUATIONS.

114. $\{\text{chd}^2(\theta+x) - \text{chd}^2(\theta-x)\}\operatorname{cosec}\theta = 2\sqrt{3}$. Find (x).

$$2\sqrt{3}\,.\sin\theta = \text{chd}^2(\theta+x) - \text{chd}^2(\theta-x)$$

$$= 2\operatorname{vers}(\theta+x) - 2\operatorname{vers}(\theta-x);$$

$$\therefore\ \sqrt{3}\,.\sin\theta = \operatorname{vers}(\theta+x) - \operatorname{vers}(\theta-x)$$

$$= \cos(\theta-x) - \cos(\theta+x)$$

$$= 2\sin\theta\,.\sin x \quad (42);$$

$$\therefore\ \sin x = \frac{\sqrt{3}}{2} = \sin 60^\circ;\quad \therefore\ x = 60^\circ.$$

Cor. The student will immediately perceive that (x) equals all the other arcs whose sines are $\frac{\sqrt{3}}{2}$.

115. $$\frac{\operatorname{vers} x}{m\operatorname{vers}(\theta-x)} = \frac{\sin x}{n\sin(\theta-x)}.\quad \text{Find } (x).$$

$$\frac{m\operatorname{vers}(\theta-x)}{\sin(\theta-x)} = \frac{n\operatorname{vers} x}{\sin x};$$

$$\therefore\ \frac{m\{1-\cos(\theta-x)\}}{\sin(\theta-x)} = \frac{n\,.(1-\cos x)}{\sin x},$$

$$\frac{2m\,.\sin^2\left(\frac{\theta-x}{2}\right)}{2\sin\left(\frac{\theta-x}{2}\right).\cos\left(\frac{\theta-x}{2}\right)} = \frac{2n\,.\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\,.\cos\frac{x}{2}};$$

$$\therefore\ m\,.\tan\left(\frac{\theta-x}{2}\right) = n\,.\tan\frac{x}{2};$$

$$\therefore\ m:n::\tan\frac{x}{2}:\tan\left(\frac{\theta-x}{2}\right);$$

$$\therefore\ m+n : m-n :: \tan\frac{x}{2}+\tan\left(\frac{\theta-x}{2}\right) : \tan\frac{x}{2}-\tan\left(\frac{\theta-x}{2}\right)$$

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$$:: \sin\left(\frac{x}{2}+\frac{\theta-x}{2}\right) : \sin\left\{\frac{x}{2}-\left(\frac{\theta-x}{2}\right)\right\} \quad (43)$$

$$:: \sin\frac{\theta}{2} : \sin\left(x-\frac{\theta}{2}\right);$$

$$\therefore\ \sin\left(x-\frac{\theta}{2}\right)=\frac{m-n}{m+n}\sin\frac{\theta}{2};$$

$$\therefore\ x-\frac{\theta}{2}=\sin^{-1}\left(\frac{m-n}{m+n}\,.\,\sin\frac{\theta}{2}\right),$$

$$\text{and}\ x=\frac{\theta}{2}+\sin^{-1}\left(\frac{m-n}{m+n}\,.\,\sin\frac{\theta}{2}\right).$$

116. $\sqrt{3}\,.\,\sin x=\sqrt{3}-\cos x.$ Find (x).

$$(1-\sin x)\sqrt{3}=\cos x;$$

$$\therefore\ 3-6\sin x+3\sin^2 x=\cos^2 x=1-\sin^2 x;$$

$$\therefore\ 4\sin^2 x-6\sin x=-2;$$

$$\therefore\ \sin^2 x-\frac{3}{2}\,.\,\sin x=-\frac{1}{2};$$

$$\therefore\ \sin^2 x-\frac{3}{2}\,.\,\sin x+\frac{9}{16}=\frac{1}{16};$$

$$\therefore\ \sin x-\frac{3}{4}=\pm\frac{1}{4};$$

$$\therefore\ \sin x=\frac{3\pm1}{4}=1\ \text{and}\ \frac{1}{2};$$

$$\therefore\ x=90\ \text{and}\ 30.$$

117. $$\frac{\sqrt{10+2\sqrt{5}}\,.\,(\sqrt{5}-1)}{16}$$

$$=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos x}}}\times\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos x}}}.$$ Find (x).

$$\frac{\sqrt{(10+2\sqrt{5})\,.\,(6-2\sqrt{5})}}{16}=\sqrt{\frac{1}{4}-\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\cos x}{2}}\right)},$$

$$\text{or } \frac{\sqrt{40-8\sqrt{5}}}{16} = \sqrt{\frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} \cdot \cos\frac{x}{2}\right)} \quad (32);$$

$$\therefore \frac{\sqrt{10-2\sqrt{5}}}{8} = \frac{1}{2}\sqrt{1 - \left(\frac{1+\cos\frac{x}{2}}{2}\right)};$$

$$\therefore \frac{\sqrt{10-2\sqrt{5}}}{4} = \sqrt{1 - \cos^2\frac{x}{4}} \quad (32)$$

$$= \sin\frac{x}{4},$$

$$\text{or } \sin 36 = \sin\frac{x}{4} \quad (22);$$

$$\therefore \frac{x}{4} = 36,$$

$$\text{and } x = 144.$$

118. $S^2 - S - 2 = 0$, where

$$S = \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \ldots\ldots}}}} \quad \text{Find } (x).$$

$$S^2 - S = 2; \qquad \therefore S^2 - S + \frac{1}{4} = \frac{9}{4};$$

$$\therefore S - \frac{1}{2} = \pm\frac{3}{2}; \qquad \therefore S = \frac{1 \pm 3}{2} = 2 \text{ and } -1.$$

First, let $S = 2$, or $\sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\ldots\ldots}}}} = 2$;

$$\therefore \sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\ldots\ldots}}}} = 4;$$

$$\therefore \sec^2 x + S = 4; \qquad \therefore \sec^2 x = 4 - S = 4 - 2 = 2;$$

$$\therefore \sec x = \pm\sqrt{2}.$$

Next, let $S = -1$;

$$\therefore \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\sec^2 x + \sqrt{\ldots\ldots}}}} = -1;$$

$$\therefore \sec^2 x + S = 1;$$

$$\therefore\ \sec^2 x = 1 - S = 1 + 1 = 2;$$

$$\therefore\ \sec x = \pm\sqrt{2}, \text{ as before;}$$

$$\therefore \text{ the values of } x \text{ are } 45 \text{ and } 135 \quad (16).$$

119. $$\sec x \,.\, \tan x = \operatorname{cosec} x \,.\, \cot x.$$

$$\text{Here } \frac{\sin x}{\cos^2 x} = \frac{\cos x}{\sin^2 x},$$

$$\text{and } \sin^3 x = \cos^3 x;$$

$$\therefore\ \cos^3 x - \sin^3 x = 0,$$

$$\text{or } (\cos^2 x + \cos x \,.\, \sin x + \sin^2 x) \,.\, (\cos x - \sin x) = 0.$$

By the solution of the quadratic factor $1 + \cos x \,.\, \sin x = 0$, we obtain two *imaginary* roots.

But from the simple factor $\cos x - \sin x = 0$, we have

$$\cos x = \sin x = \cos\left(\frac{\pi}{2} - x\right);$$

$$\therefore\ x = \frac{\pi}{2} - x,$$

$$\text{and } x = 45.$$

120. $\cot x \,.\, \tan 2x - \tan x \,.\, \cot 2x = 2$. Find the values of (x).

Let $\tan 2x \,.\, \cot x = y$; then $\tan x \,.\, \cot 2x = \frac{1}{y}$ (23);

$$\therefore\ y - \frac{1}{y} = 2, \quad \text{and } y^2 - 2y = 1;$$

$$\therefore\ y^2 - 2y + 1 = 2;$$

$$\therefore\ y - 1 = \pm\sqrt{2};$$

$$\therefore\ y = 1 \pm \sqrt{2}.$$

And since $$\tan 2x \,.\, \cot x = 1 + \sqrt{2},$$

$$\frac{2}{1 - \tan^2 x} = 1 + \sqrt{2} \quad (33);$$

$$\therefore\ 2 = 1 + \sqrt{2} - (1 + \sqrt{2}) \,.\, \tan^2 x;$$

$$\therefore\ (1+\sqrt{2})\,.\,\tan^2 x=\sqrt{2}-1\,;$$

$$\therefore\ \tan^2 x=\frac{\sqrt{2}-1}{\sqrt{2}+1}=(\sqrt{2}-2)^2\,;$$

$$\therefore\ \tan x=\pm(\sqrt{2}-1).$$

Also, since $\tan 2x\,.\,\cot x=1-\sqrt{2}$, by proceeding in the same manner we obtain

$$\tan x=\pm(\sqrt{2}+1)\,;$$

$\therefore$ the 4 values of x are $22^\circ.30'$; $157^\circ.30'$; $67^\circ.30'$; $112^\circ.30'$ (15) and (5).

121. $\cot(\pi-3x)=\tan(x-\pi)$. Find values of (x).

$$\text{Here } -\cot 3x=-\tan(\pi-x)\,;$$

$$\therefore\ \cot 3x=\tan(\pi-x).$$

$$\text{Let } \tan x=y.$$

$$\text{Then } \frac{3y^2-1}{y^3-3y}=-y \quad (35),\ (23) \text{ and } (5)\,;$$

$$\therefore\ 3y^2-1=3y^2-y^4\,;$$

$$\therefore\ y^4-1=0,$$

$$\text{or } (y^2+1)\,.\,(y^2-1)=0.$$

Taking the quadratic factor $y^2+1=0$, we obtain two *imaginary* roots.

But .... $y^2-1=0$, we obtain $y=\pm 1$;

$$\therefore\ x=45 \text{ and } 135 \quad (12).$$

122. $\cot^3(\varpi-x)=\dfrac{1-3\cot^2 x}{1-3\tan^2 x}$. Find the values of (x).

$$\text{Let } \tan x=y.$$

$$\text{Then } -\frac{1}{y^3}=\frac{1-\dfrac{3}{y^2}}{1-3y^2} \quad (6) \text{ and } (23)\,;$$

$$\therefore\ 3y^2-1=y^3-3y,$$

$$\text{and } y^3+1-3y\,.\,(y+1)=0,$$

$$\text{or } \{(y^2-y+1)-3y\}\,.\,(y+1)=0.$$

Taking the quadratic factor $y^2 - 4y + 1 = 0$, we obtain

$$y^2 - 4y = -1;$$

$$\therefore\ y^2 - 4y + 4 = 3;$$

$$\therefore\ y - 2 = \pm\sqrt{3} \text{ and } y = 2 \pm \sqrt{3},$$

$$\text{and } x = 75 \text{ and } 15 \quad (21).$$

From the factor $y + 1 = 0$, we have

$$y = -1 \text{ and } x = 135.$$

123. $2\sin^2\left(\frac{\pi}{2} - \frac{3x}{2}\right) = \text{chd}^2 x - \text{vers}^2 x$. Find values of $\cos x$.

Here $1 - \cos(\pi - 2x) = 2\,\text{vers}\,x - \text{vers}^2 x = \text{vers}\,x\,(2 - \text{vers}\,x)$,

or $1 + \cos 3x = (1 - \cos x)(1 + \cos x) = 1 - \cos^2 x$.

Let $\cos x = y$.

$$\text{Then } 1 + 4y^3 - 3y = 1 - y^2 \quad (34);$$

$$\therefore\ (4y^2 + y - 3)y = 0;$$

$$\therefore\ 4y^2 + y - 3 = 0 \text{ and } y = 0;$$

$$\therefore\ 4y^2 + y = 3;$$

$$\therefore\ 64y^2 + 16y + 1 = 49;$$

$$\therefore\ 8y + 1 = \pm 7;$$

$$\therefore\ 8y = -1 \pm 7 = 6 \text{ and } -8;$$

$$\therefore\ y = \tfrac{3}{4} \text{ and } -1,$$

so that the 3 values of $\cos x$ are 0, $\frac{3}{4}$ and -1.

124. $$\frac{\cot\frac{x}{2}}{\text{chd}\,x} = \sqrt{5}. \quad \text{Find } \cos x.$$

Here $\sqrt{\left(\frac{1+\cos x}{1-\cos x}\right)} \cdot \frac{1}{\sqrt{(2 - 2\cos x)}} = \sqrt{5}$ (25) and (31).

Let $\cos x = y$.

$$\text{Then } \frac{1+y}{(1-y)^2}=10,$$

$$\text{and } 1+y=10-20y+10y^2;$$

$$\therefore\ 10y^2-21y=-9,$$

$$y^2-\frac{21}{10}y+\frac{441}{400}=\frac{81}{400};$$

$$\therefore\ y-\frac{21}{20}=\frac{\pm 9}{20};$$

$$\therefore\ y=\frac{21\pm 9}{20}=1\frac{1}{2} \text{ and } \frac{3}{5}.$$

But the cosine of an angle is never greater than the radius;

$$\therefore\ \cos x=\frac{3}{5}.$$

125. $$\frac{\cot x-\sec x}{\cot x}=\frac{1}{16}.\quad \text{Find } \tan x.$$

$$\text{Here } 1-\sec x\,.\,\tan x=\frac{1}{16};$$

$$\therefore\ \sec x\,.\,\tan x=\frac{15}{16},$$

$$\text{or } (1+\tan^2 x)\,.\,\tan^2 x=\frac{225}{256}.$$

Let $\tan x=y$.

$$\text{Then } y^4+y^2=\frac{225}{256},$$

$$\text{and } y^4+y^2+\frac{1}{4}=\frac{289}{256};$$

$$\therefore\ y^2+\frac{1}{2}=\frac{\pm 17}{16};$$

$$\therefore\ y^2=\frac{-8\pm 17}{16}=\frac{9}{16} \text{ and } \frac{-25}{16}.$$

Rejecting therefore the negative value as giving two *imaginary* roots we have $y=\pm\frac{3}{4}$ or $\tan x=\pm\frac{3}{4}$.

126. $z + z^{-1} = \sqrt{(\operatorname{cosec}^2\theta + \sec^2\theta)}$. Find values of z.

$$\operatorname{cosec}^2\theta + \sec^2\theta = \cot^2\theta + 1 + \tan^2\theta + 1 = \cot^2\theta + 2\cot\theta\,.\tan\theta + \tan^2\theta \quad (23);$$

$$\therefore\ z + \frac{1}{z} = \sqrt{(\operatorname{cosec}^2\theta + \sec^2\theta)} = \cot\theta + \tan\theta = \tan\theta + \frac{1}{\tan\theta};$$

$$\therefore\ z - \tan\theta = \frac{1}{\tan\theta} - \frac{1}{z} = \frac{z - \tan\theta}{z\,.\tan\theta};$$

$$\therefore\ z\tan\theta(z - \tan\theta) = z - \tan\theta;$$

$$\therefore\ (z\tan\theta - 1)\,.(z - \tan\theta) = 0;$$

$$\therefore\ z\tan\theta - 1 = 0, \quad \text{and } z = \cot\theta.$$

$$\text{Also } z - \tan\theta = 0, \quad \text{and } z = \tan\theta.$$

127. $\dfrac{x\,.\sqrt{(4 - x^2)}}{2 - x^2} + \dfrac{2 - x^2}{x\,.\sqrt{(4 - x^2)}} = \sec\theta\,.\operatorname{cosec}\theta$. Find the values of (x).

$$\text{Let } \frac{x\,.\sqrt{(4 - x^2)}}{2 - x^2} = z. \quad \text{Then } \frac{2 - x^2}{x\,.\sqrt{(4 - x^2)}} = \frac{1}{z},$$

$$\text{and } z + \frac{1}{z} = \sec\theta\,.\operatorname{cosec}\theta$$

$$= \frac{2}{2\sin\theta\,.\cos\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec} 2\theta$$

$$= \cot\theta + \tan\theta \quad (27);$$

$\therefore$ the values of (z) are $\cot\theta$ and $\tan\theta$. (Prop. 126).

$$\text{Let } \frac{x\,.\sqrt{(4 - x^2)}}{2 - x^2} = \cot\theta.$$

$$\text{Then } 4x^2 - x^4 = 4\cot^2\theta - (4x^2 - x^4)\,.\cot^2\theta;$$

$$\therefore\ 4\cot^2\theta = (1 + \cot^2\theta)\,.(4x^2 - x^4) = \operatorname{cosec}^2\theta\,.(4x - x^4);$$

$$\therefore\ 4x^2 - x^4 = \frac{4\cot^2\theta}{\operatorname{cosec}^2\theta} = 4\cos^2\theta \quad (25);$$

$$\therefore\ x^4 - 4x^2 = -4\cos^2\theta;$$

$$\therefore\ x^4-4x^2+4=4(1-\cos^2\theta)=4\sin^2\theta;$$

$$\therefore\ x^2-2=\pm 2\sin\theta;$$

$$\therefore\ x^2=2\pm 2\sin\theta=2(1\pm\sin\theta);$$

$$\therefore\ x=\pm\sqrt{2}\,.\sqrt{1\pm\sin\theta}.$$

Similarly when $\frac{x\,.\sqrt{(4-x^2)}}{2-x^2}=\tan\theta$, it may be shewn that

$$x=\pm\sqrt{2}\,.\sqrt{1\pm\cos\theta}.$$

128. Tan $3\theta=\sin 6\theta$. Find values of θ.

Let $3\theta=\phi$.

$$\text{Then }\tan\phi=\sin 2\phi=\frac{2\tan\phi}{1+\tan^2\phi}\quad(29);$$

$$\therefore\ \tan\phi+\tan^3\phi=2\tan\phi;$$

$$\therefore\ (\tan^2\phi-1)\,.\tan\phi=0;$$

$$\therefore\ \tan\phi=0,\quad\text{and}\quad\tan^2\phi-1=0;$$

$\therefore$ the 3 values of $\tan\phi$ are 0, 1 and -1.

Case 1. Let $\tan\phi=0$; $\therefore\ \tan 3\theta=0$.

Let $\tan\theta=x$.

Then $\frac{x^3-3x}{3x^2-1}=0$ (35); $\therefore\ x^3-3x=0$; $\therefore\ (x^2-3)\,.x=0$;

$$\therefore\ x^2-3=0,\quad\text{and}\quad x=0;$$

$\therefore$ 3 values of $\tan\theta$ are 0, $\sqrt{3}$ and $-\sqrt{3}$,

and the corresponding values of θ are 0, 60 and 120 (19).

Case 2. Again, let $\tan\phi=1$;

$$\therefore\ x^3-3x=3x^2-1;$$

$$\therefore\ (x^3+1)-3x\,.(x+1)=0;$$

$$\therefore\ \{(x^2-x+1)-3x\}\,.(x+1)=0,$$

$$\text{or}\ (x^2-4x+1)\,.(x+1)=0;$$

from the solution of which factors we obtain 3 other values of $\tan\theta$; -1, $2+\sqrt{3}$ and $2-\sqrt{3}$; and corresponding values of θ are 135, 75 and 15 (21).

CASE 3. Also, let $\tan\phi = -1$;

$$\therefore\ x^3 - 3x = 1 - 3x^2;$$

$$\therefore\ (x^3 - 1) + 3x.(x-1) = 0,$$

$$\text{or } \{(x^2 + x + 1) + 3x\}.(x-1) = 0;$$

$$\therefore\ (x^2 + 4x + 1).(x-1) = 0;$$

from the solution of which factors we get the 3 remaining values of $\tan\theta$; 1, $\sqrt{3} - 2$, $-2 - \sqrt{3}$; and the 3 corresponding values of θ are 45, 165, 105;

$\therefore$ θ has 9 values, and these are

0, 15, 45, 60, 75, 105, 120, 135, 165.

129. $\dfrac{\tan\theta}{\sec\left(\dfrac{\pi}{3} - \theta\right)} = \frac{1}{2}$. Find values of $\sin\theta$.

$$\tfrac{1}{2} = \frac{\sin\theta}{\cos\theta} \cdot \cos\left(\frac{\pi}{3} - \theta\right)$$

$$= \frac{\sin\theta}{\cos\theta}\left(\cos\frac{\pi}{3} \cdot \cos\theta + \sin\frac{\pi}{3} \cdot \sin\theta\right) = \tfrac{1}{2}\sin\theta + \frac{\sqrt{3}}{2} \cdot \frac{\sin^2\theta}{\cos\theta};$$

$$\therefore\ 1 = \sin\theta + \frac{\sqrt{3} \cdot \sin^2\theta}{\sqrt{(1 - \sin^2\theta)}}.$$

Let $\sin\theta = x$.

$$\text{Then } \sqrt{(1 - x^2)} = x \cdot \sqrt{(1 - x^2)} + \sqrt{3} \cdot x^2;$$

$$\therefore\ \sqrt{3} \cdot x^2 = (1 - x) \cdot \sqrt{(1 - x^2)};$$

$$\therefore\ 3x^4 = (1 - 2x + x^2)(1 - x^2)$$

$$= (1 + x^2) \cdot (1 - x^2) - 2x(1 - x^2) = 1 - x^4 - 2x + 2x^3;$$

$$\therefore\ 4x^4 - 2x^3 + 2x - 1 = 0,$$

$$\text{or } 2x^3 (2x - 1) + (2x - 1) = 0,$$

$$\text{or } (2x^3 + 1)(2x - 1) = 0;$$

$$\therefore\ (x^3 + \tfrac{1}{2})(x - \tfrac{1}{2}) = 0,$$

$$\text{or } [x^3 + \{\sqrt[3]{(\tfrac{1}{2})}\}^3](x - \tfrac{1}{2}) = 0,$$

$$\text{or } \{x^2 - \sqrt[3]{(\tfrac{1}{2})}\,x + (\sqrt[3]{\tfrac{1}{2}})^2\}\,(x + \sqrt[3]{\tfrac{1}{2}})(x - \tfrac{1}{2}) = 0.$$

From the solution of the quadratic factor we obtain two *imaginary* roots, but from the two simple factors we have

$$x = -\sqrt[3]{(\tfrac{1}{2})} \text{ and } x = \tfrac{1}{2} \text{ the required values of } \sin\theta.$$

130. $\operatorname{cosec}(75-\theta) - \operatorname{cosec} 270 = \operatorname{cosec}(75+\theta) + \operatorname{cosec} 60$. Find $\sin\theta$.

$$\text{Here } \frac{1}{\sin(75-\theta)} - \frac{1}{\sin(75+\theta)} = \frac{2}{\sqrt{3}} - 1 \quad (12) \text{ and } (18);$$

$$\therefore \frac{\sin(75+\theta) - \sin(75-\theta)}{\sin(75+\theta) \,.\, \sin(75-\theta)} = \frac{2-\sqrt{3}}{\sqrt{3}};$$

$$\therefore \frac{2\cos 75 \,.\, \sin\theta}{\sin^2 75 - \sin^2\theta} = \frac{2-\sqrt{3}}{\sqrt{3}} \quad (41) \text{ and } (42);$$

$$\therefore \frac{2-\sqrt{3}}{\sqrt{3}} \,.\, \sin^2 75 - \frac{2-\sqrt{3}}{\sqrt{3}} \,.\, \sin^2\theta = 2\cos 75 \,.\, \sin\theta;$$

$$\therefore \frac{2-\sqrt{3}}{\sqrt{3}} \sin^2\theta + 2\cos 75 \,.\, \sin\theta = \frac{2-\sqrt{3}}{\sqrt{3}} \,.\, \sin^2 75;$$

$$\therefore \sin^2\theta + \frac{2\,.\,(\sqrt{3}-1)}{2\sqrt{2}} \,.\, \frac{\sqrt{3}}{2-\sqrt{3}} \,.\, \sin\theta = \frac{2-\sqrt{3}}{\sqrt{3}} \,.\, \frac{\sqrt{3}}{2-\sqrt{3}} \,.\, \left(\frac{4+2\sqrt{3}}{8}\right) \quad (21);$$

$$\therefore \sin^2\theta + \frac{\sqrt{6}}{\sqrt{3}-1} \,.\, \sin\theta = \frac{2+\sqrt{3}}{4};$$

$$\begin{aligned}
\therefore \sin^2\theta + \frac{\sqrt{6}}{\sqrt{3}-1} \,.\, \sin\theta + \frac{3}{2\,.\,(4-2\sqrt{3})} &= \frac{2+\sqrt{3}}{4} + \frac{3}{4\,.\,(2-\sqrt{3})} \\
&= \tfrac{1}{4}\left(2 + \sqrt{3} + \frac{3}{2-\sqrt{3}}\right) \\
&= \frac{1}{2-\sqrt{3}} \\
&= \frac{2}{4-2\sqrt{3}};
\end{aligned}$$

$$\therefore \sin\theta + \frac{\sqrt{3}}{\sqrt{2}\,.\,(\sqrt{3}-1)} = \frac{\pm\sqrt{2}}{\sqrt{3}-1};$$

$$\therefore \sin\theta = \frac{-\sqrt{3} \pm 2}{\sqrt{2}(\sqrt{3}-1)}.$$

131.
$$\frac{n\tan^2\theta + n\sin^2\theta}{\frac{m}{2}\tan^2\theta + \frac{n}{2}\sin^2\theta - \frac{n}{2}\tan^2\theta - \frac{m}{2}\sin^2\theta}$$

$$= m\operatorname{cosec}^2\theta + n\cot^2\theta + n\operatorname{cosec}^2\theta + m\cot^2\theta.$$

Find (n) in terms of (m).

Here $\dfrac{2n\,.\,(\tan^2\theta + \sin^2\theta)}{m(\tan^2\theta - \sin^2\theta) - n(\tan^2\theta - \sin^2\theta)}$

$$= m(\operatorname{cosec}^2\theta + \cot^2\theta) + n(\operatorname{cosec}^2\theta + \cot^2\theta),$$

or $\dfrac{2n(\tan^2\theta + \sin^2\theta)}{(m-n)\tan^2\theta - \sin^2\theta)} = (m+n)\,.\,(\operatorname{cosec}^2\theta + \cot^2\theta);$

$$\therefore \frac{2n}{(m+n)(m-n)} = \frac{(\operatorname{cosec}^2\theta + \cot^2\theta)\,.\,(\tan^2\theta - \sin^2\theta)}{\tan^2\theta + \sin^2\theta}$$

$$= \frac{\left(\frac{1}{\sin^2\theta} + \frac{1}{\tan^2\theta}\right)\,.\,(\tan^2\theta - \sin^2\theta)}{\tan^2\theta + \sin^2\theta}$$

$$= \frac{\tan^2\theta - \sin^2\theta}{\tan^2\theta\,.\,\sin^2\theta}$$

$$= \frac{1}{\sin^2\theta} - \frac{1}{\tan^2\theta}$$

$$= \operatorname{cosec}^2\theta - \cot^2\theta$$

$$= 1;$$

$$\therefore\ 2n = (m+n)(m-n) = m^2 - n^2;$$

$$\therefore\ n^2 + 2n = m^2,$$

$$n^2 + 2n + 1 = m^2 + 1$$

$$n + 1 = \pm\sqrt{(m^2+1)} \text{ and } n = -1 \pm \sqrt{(m^2+1)}.$$

132. $\sin n\theta = \dfrac{x^n - x^{-n}}{2\sqrt{-1}}$ and $\cos m\theta = \dfrac{x^{3n} + x^{-3n}}{2}$. Eliminate (x) and (θ).

Here $\sin^2 n\theta = \dfrac{x^{2n} - 2 + x^{-2n}}{-4} = \dfrac{2 - (x^{2n} + x^{-2n})}{4};$

$$\therefore\ \cos^2 n\theta = 1 - \left\{\frac{2-(x^{2n}+x^{-2n})}{4}\right\} = \frac{x^{2n}+2+x^{-2n}}{4};$$

$$\therefore\ \cos n\theta = \frac{x^n+x^{-n}}{2};$$

$$\therefore\ \cos^3 n\theta = \frac{x^{3n}+3x^n+3x^{-n}+x^{-3n}}{8} = \frac{x^{3n}+x^{-3n}}{8} + 3\,.\left(\frac{x^n+x^{-n}}{8}\right);$$

$$\begin{aligned}\therefore\ \cos 3n\theta &= 4\cos^3 n\theta - 3\cos n\theta \quad (34)\\ &= \frac{x^{3n}+x^{-3n}}{2} + 3\,.\left(\frac{x^n+x^{-n}}{2}\right) - 3\,.\left(\frac{x^n+x^{-n}}{2}\right)\\ &= \frac{x^{3n}+x^{-3n}}{2}\\ &= \cos m\,.\,\theta;\end{aligned}$$

$$\therefore\ 3n\theta = m\theta;$$

$$\therefore\ 3n = m.$$

133. Eliminate θ by means of the equations $n^2\tan^2\theta - n\sec^2\theta = a$ and $m^2\cot^2\theta - m\operatorname{cosec}^2\theta = b$. Find also the values of (θ) when

$$m = \frac{1+a-b}{2} \text{ and } n = \frac{1+b-a}{2}.$$

Here $n\sec^2\theta - n^2\tan^2\theta = -a$.

But $n\sec^2\theta - n\tan^2\theta = n$ (23);

$$\therefore\ (n^2-n)\tan^2\theta = a+n; \qquad \therefore\ \tan^2\theta = \frac{a+n}{n^2-n}.$$

Again $m\operatorname{cosec}^2\theta - m^2\cot^2\theta = -b$.

But $m\operatorname{cosec}^2\theta - m\cot^2\theta = m$;

$$\therefore\ (m^2-m)\cot^2\theta = b+m; \qquad \therefore\ \cot^2\theta = \frac{b+m}{m^2-m};$$

$$\therefore\ \tan^2\theta = \frac{1}{\cot^2\theta} = \frac{m^2-m}{b+m};$$

$$\therefore\ \frac{a+n}{n^2-n} = \frac{m^2-m}{b+m}.$$

Again $m = \frac{1+a-b}{2} = \frac{1}{2} + \frac{a-b}{2}$,

$$n = \frac{1+b-a}{2} = \frac{1}{2} - \frac{a-b}{2};$$

$$\therefore\ m + n = 1;$$

$$\therefore\ m - n = (m+n)(m-n) = m^2 - n^2,$$

$$\text{and } m^2 - m = n^2 - n.$$

Also $m - n = a - b$;

$$\therefore\ a + n = b + m;$$

$$\therefore\ \tan^2\theta = \frac{a+n}{n^2-n} = \frac{b+m}{m^2-m} = \cot^2\theta;$$

$$\therefore\ \tan\theta = \pm\cot\theta = \pm\frac{1}{\tan\theta};$$

$$\therefore\ \tan^2\theta = \pm 1;$$

$$\therefore\ \tan\theta = \pm\sqrt{-1} \text{ and } \pm 1.$$

So that rejecting the two *imaginary* values, we have

$$\tan\theta = \pm 1,$$

$$\text{and } \theta = 45 \text{ and } 135.$$

134. Solve the equation $x^4 - 2\cos 2\theta x^2 + 1 = 0$, and shew that its roots are of the form $\pm a$, $\pm\frac{1}{a}$.

Here $x^4 - 2\cos 2\theta x^2 = -1$

$$x^4 - 2\cos 2\theta \,.\, x^2 + \cos^2 2\theta = \cos^2 2\theta - 1 = -1 \,.\, \sin^2 2\theta;$$

$$\therefore\ x^2 - \cos 2\theta = \pm\sqrt{-1} \,.\, \sin 2\theta;$$

$$\therefore\ x^2 = \cos 2\theta \pm \sqrt{-1} \,.\, \sin 2\theta$$

$$= (\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^2 \quad (56);$$

$$\therefore\ x = \pm(\cos\theta + \sqrt{-1} \,.\, \sin\theta) \text{ and } \pm(\cos\theta - \sqrt{-1} \,.\, \sin\theta).$$

$$\text{But } \cos\theta - \sqrt{-1}\,.\,\sin\theta = \frac{\cos\theta - \sqrt{-1}\,.\,\sin\theta}{\cos^2\theta + \sin^2\theta} \quad (23)$$

$$= \frac{\cos\theta - \sqrt{-1}\,.\,\sin\theta}{(\cos\theta + \sqrt{-1}\,.\,\sin\theta)\,.\,(\cos\theta - \sqrt{-1}\,.\,\sin\theta)}$$

$$= \frac{1}{\cos\theta + \sqrt{-1}\,.\,\sin\theta};$$

therefore the roots are

$$\pm(\cos\theta + \sqrt{-1}\,.\,\sin\theta) \text{ and } \pm\frac{1}{(\cos\theta + \sqrt{-1}\,.\,\sin\theta)},$$

and therefore are of the form $\pm a$ and $\pm\dfrac{1}{a}$.

135. $\log_\epsilon(\cos 3\theta \pm \sqrt{-1}\,.\,\sin 3\theta)$

$$= 2\log_\epsilon(\cos x\theta \pm \sqrt{-1}\,.\,\sin x\theta). \quad \text{Find } (x).$$

$$\log_\epsilon(\cos 3\theta \pm \sqrt{-1}\,.\,\sin 3\theta)$$

$$= \log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta)^3 \quad (56)$$

$$= 3\log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta) \quad (71).$$

Also

$$2\log_\epsilon(\cos x\theta \pm \sqrt{-1}\,.\,\sin x\theta) = 2\log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta)^x \quad (56)$$

$$= 2x\,.\,\log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta) \quad (71);$$

$$\therefore\ 3\log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta) = 2x\,.\,\log_\epsilon(\cos\theta \pm \sqrt{-1}\,.\,\sin\theta);$$

$$\therefore\ 2x = 3 \text{ and } x = \frac{3}{2} = 1\frac{1}{2}.$$

136. If $\dfrac{1}{2\epsilon^{-\frac{\theta}{2}\sqrt{-1}}} + \dfrac{1}{2\epsilon^{\frac{\theta}{2}\sqrt{-1}}} = \dfrac{1}{\cot\dfrac{\theta}{2} - \tan\dfrac{\theta}{2}}$. Then $\theta = 60°$.

$$\text{Here } \frac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} + \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{2} = \frac{1}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}};$$

$$\therefore\ \cos\frac{\theta}{2} = \frac{1}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}};$$

$$\therefore\ \frac{\cos^2\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} - \sin\frac{\theta}{2} = 1\,;$$

$$\therefore\ \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \sin\frac{\theta}{2}\,;$$

$$\therefore\ \cos\theta = \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right);$$

$$\therefore\ \theta = \frac{\pi}{2} - \frac{\theta}{2}\,;$$

$$\therefore\ \frac{3\theta}{2} = \frac{\pi}{2}\,;$$

$$\therefore\ 3\theta = \pi, \quad \text{and}\ \ \theta = \frac{\pi}{3} = 60^\circ.$$

137. $\dfrac{\epsilon^{\frac{\text{chd}\,\theta}{2}}}{\epsilon^{\text{chd}\,\theta}} + \dfrac{\epsilon^{\text{chd}\,\theta}}{\epsilon^{\frac{\text{chd}\,\theta}{2}}} = \epsilon^{\frac{3\,\text{chd}\,\theta}{2}}$. Find chd θ.

$$\text{Let}\ \epsilon^{\text{chd}\,\theta} = y.$$

$$\text{Then}\ \epsilon^{\frac{\text{chd}\,\theta}{2}} = y^{\frac{1}{2}}, \quad \text{and}\ \ \epsilon^{\frac{3\,\text{chd}\,\theta}{2}} = y^{\frac{3}{2}}\,;$$

$$\therefore\ \frac{y^{\frac{1}{2}}}{y} + \frac{y}{y^{\frac{1}{2}}} = y^{\frac{3}{2}},$$

$$\therefore\ 1 + y = y^2\,;$$

$$\therefore\ y^2 - y = 1\,;$$

$$\therefore\ y^2 - y + \frac{1}{4} = \frac{5}{4}\,;$$

$$\therefore\ y - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}\,;$$

$$\therefore\ y = \frac{1 \pm \sqrt{5}}{2},$$

$$\text{or}\ \ \epsilon^{\text{chd}\,\theta} = \frac{1 \pm \sqrt{5}}{2}\,;$$

$$\therefore\ \log_\epsilon (\epsilon^{\operatorname{chd}\theta}) = \log_\epsilon \left(\frac{1 \pm \sqrt{5}}{2}\right);$$

$$\therefore\ \operatorname{chd}\theta = \log_\epsilon (1 \pm \sqrt{5}) - \log_\epsilon 2 \quad (73) \text{ and } (72).$$

138. $(1 + \cos\theta) . \cos^4 x + (2\cos\theta - 6) . \cos^2 x + 1 + \cos\theta = 0.$ Find values of $\cos x$.

Let $\cos x = y$.

$$\text{Then } y^4 + \frac{2\cos\theta - 6}{1 + \cos\theta} . y^2 = -1.$$

$$\text{But } \frac{2\cos\theta - 6}{1 + \cos\theta} = \frac{2\cos\theta - 6}{2\cos^2\frac{\theta}{2}} = \frac{\cos\theta - 3}{\cos^2\frac{\theta}{2}}$$

$$= \frac{1 - 2\sin^2\frac{\theta}{2} - 3}{\cos^2\frac{\theta}{2}} = -2 . \left(\frac{1 + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)$$

$$= -2 . \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right);$$

$$\therefore\ y^4 - 2 . \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right) . y^2 = -1;$$

$$\therefore\ y^4 - 2 . \left(\sec\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right) . y^2 + \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right)^2 = \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right)^2 - 1$$

$$= \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2} + 1\right) . \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2} - 1\right)$$

$$= 2\sec^2\frac{\theta}{2} . 2\tan^2\frac{\theta}{2} \quad (23)$$

$$= 4\sec^2\frac{\theta}{2} . \tan^2\frac{\theta}{2};$$

$$\therefore\ y^2 - \left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\right) = \pm 2\sec\frac{\theta}{2} . \tan\frac{\theta}{2};$$

$$\therefore\ y^2 = \sec^2\frac{\theta}{2} \pm 2\sec\frac{\theta}{2} . \tan\frac{\theta}{2} + \tan^2\frac{\theta}{2},$$

$$\text{and } y = \pm\left(\sec\frac{\theta}{2} \pm \tan\frac{\theta}{2}\right);$$

$\therefore$ the 4 values of $\cos x$ are,

$$\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\ldots\ldots(1),$$

$$\sec\frac{\theta}{2}-\tan\frac{\theta}{2}\ldots\ldots(2),$$

$$-\sec\frac{\theta}{2}-\tan\frac{\theta}{2}\ldots\ldots(3),$$

$$-\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\ldots\ldots(4).$$

139. Form the equation whose roots are

$$\pm p\,.\cos^2\frac{\theta}{2};\ \pm p\,.\sin^2\frac{\theta}{2},\ \text{where } \sin\theta=\frac{2\sqrt{q}}{p}.$$

Let x be the quantity whose four values are

$$\pm p\,.\cos^2\frac{\theta}{2},\ \pm p\,.\sin^2\frac{\theta}{2}.$$

Then the four factors of the required equation will evidently be

$$x-p\,.\cos^2\frac{\theta}{2}=0;\ x+p\,.\cos^2\frac{\theta}{2}=0;\ x-p\,.\sin^2\frac{\theta}{2}=0;\ x+p\,.\sin^2\frac{\theta}{2}=0,$$

and therefore the equation itself is

$$\left(x-p\,.\cos^2\frac{\theta}{2}\right).\left(x+p\,.\cos^2\frac{\theta}{2}\right).\left(x-p\,.\sin^2\frac{\theta}{2}\right).\left(x+p\,.\sin^2\frac{\theta}{2}\right)=0,$$

$$\text{or } \left(x^2-p^2\,.\cos^4\frac{\theta}{2}\right).\left(x^2-p^2\,.\sin^4\frac{\theta}{2}\right)=0,$$

$$\text{or } x^4-p^2\,.\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right)x^2+p^4\,.\sin^4\frac{\theta}{2}\,.\cos^4\frac{\theta}{2}=0.$$

$$\text{Now } \cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}=\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)^2-2\sin^2\frac{\theta}{2}\,.\cos^2\frac{\theta}{2}$$

$$=1-2\,.\left(\frac{\sin\theta}{2}\right)^2$$

$$=1-\frac{\sin^2\theta}{2}$$

$$=1-\frac{2q}{p^2}$$

$$=\frac{p^2-2q}{p^2};$$

$$\therefore\ p^2\,.\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right)=p^2-2q.$$

$$\text{Again } \sin^4\frac{\theta}{2} \cdot \cos^4\frac{\theta}{2} = \left(\frac{\sin\theta}{2}\right)^4$$

$$= \left(\frac{\sqrt{q}}{p}\right)^4$$

$$= \frac{q^2}{p^4};$$

$$\therefore\ p^4 \cdot \sin^4\frac{\theta}{2} \cdot \cos^4\frac{\theta}{2} = q^2;$$

therefore the equation required will be

$$x^4 - (p^2 - 2q)\,x^2 + q^2 = 0.$$

140. Given the roots of $x^2 - px + q = 0$; $\frac{p}{2}(1 \pm \sqrt{-1} \cdot \tan\theta)$ to shew that $\cos\theta = \frac{p}{2\sqrt{q}}$.

$$x^2 - px + q = 0 = \left\{x - \frac{p}{2}(1 + \sqrt{-1} \cdot \tan\theta)\right\} \cdot \left\{x - \frac{p}{2}(1 - \sqrt{-1} \cdot \tan\theta)\right\}$$

$$= \left\{\left(x - \frac{p}{2}\right) - \frac{p}{2}\sqrt{-1} \cdot \tan\theta\right\} \cdot \left\{\left(x - \frac{p}{2}\right) + \frac{p}{2}\sqrt{-1} \cdot \tan\theta\right\}$$

$$= \left(x - \frac{p}{2}\right)^2 + \frac{p^2}{4}\tan^2\theta$$

$$= x^2 - px + \frac{p^2}{4} + \frac{p^2}{4} \cdot \tan^2\theta$$

$$= x^2 - px + \frac{p^2}{4} \cdot (1 + \tan^2\theta)$$

$$= x^2 - px + \frac{p^2}{4} \cdot \sec^2\theta;$$

$$\therefore\ q = \frac{p^2}{4} \cdot \sec^2\theta;$$

$$\therefore\ \sec^2\theta = \frac{4q}{p^2};$$

$$\therefore\ \cos^2\theta = \frac{1}{\sec^2\theta} = \frac{p^2}{4q},$$

$$\text{and } \cos\theta = \frac{p}{2\sqrt{q}}.$$

141. Solve the equation $x^4-(p^2+2q)x^2+q^2=0$, by a subsidiary angle,

$$x^4-(p^2+2q)x^2=-q^2,$$

$$x^4-(p^2+2q)x^2+\frac{(p^2+2q)^2}{4}=\frac{(p^2+2q)^2-4q^2}{4}=\frac{p^4+4qp^2}{4};$$

$$\therefore\ x^2-\left(\frac{p^2+2q}{2}\right)=\pm\frac{\sqrt{(p^4+4qp^2)}}{2};$$

$$\therefore\ x^2=\frac{p^2+2q\pm\sqrt{(p^4+4qp^2)}}{2}$$

$$=\frac{p^2}{2}+q\pm p\cdot\sqrt{\left(\frac{p^2}{4}+q\right)}$$

$$=\frac{p^2}{4}+q\pm p\cdot\sqrt{\left(\frac{p^2}{4}+q\right)}+\frac{p^2}{4}$$

$$=\left\{\sqrt{\left(\frac{p^2}{4}+q\right)}\pm\frac{p}{2}\right\}^2;$$

$$\therefore\ x=\pm\left\{\sqrt{\left(\frac{p^2}{4}+q\right)}\pm\frac{p}{2}\right\}$$

$$=\pm\sqrt{q}\left\{\sqrt{\left(\frac{p^2}{4q}+1\right)}\pm\frac{p}{2\sqrt{q}}\right\}.$$

Let $\frac{p}{2\sqrt{q}}=\cot 2\theta$.

Then $\frac{p^2}{4q}+1=1+\cot^2 2\theta=\operatorname{cosec}^2 2\theta$;

$$\therefore\ x=\pm\sqrt{q}\,(\operatorname{cosec} 2\theta\pm\cot 2\theta)$$

$$=\pm\sqrt{q}\,.\cot\theta \text{ and } \pm\sqrt{q}\,.\tan\theta \quad (28).$$

142. $\sqrt{a^4-a^2b^2}\,.\sin x+ab\,.\cos x-2b\,.\sqrt{(a^2-b^2)}=0$. Find (x).

$$a^2\,.\sqrt{\left(1-\frac{b^2}{a^2}\right)}\,.\sin x+ab\,.\cos x=2ab\,.\sqrt{\left(1-\frac{b^2}{a^2}\right)};$$

$$\therefore\ a\,.\sqrt{\left(1-\frac{b^2}{a^2}\right)}\,.\sin x+b\,.\cos x=2b\,.\sqrt{\left(1-\frac{b^2}{a^2}\right)}.$$

$$\text{Let } \frac{b}{a} = \sin\theta, \text{ or } b = a \,.\, \sin\theta.$$

$$\text{Then } a \,.\, \surd(1 - \sin^2\theta) \,.\, \sin x + b \,.\, \cos x = 2b \,.\, \surd(1 - \sin^2\theta),$$

$$\text{and } a \,.\, \cos\theta \,.\, \sin x + a \sin\theta \,.\, \cos x = 2a \,.\, \sin\theta \,.\, \cos\theta;$$

$$\therefore \cos\theta \,.\, \sin x + \sin\theta \,.\, \cos x = 2\sin\theta \,.\, \cos\theta;$$

$$\therefore \sin(x + \theta) = \sin 2\theta;$$

$$\therefore x + \theta = 2\theta,$$

$$\text{and } x = \theta,$$

which is known from the equation, $\log\sin\theta = \log_{10} b - \log_{10} a + 10$ to tabular radius (72).

143. If $(\operatorname{cosec} 2\theta - \cot 2\theta)\cos^2\theta = \frac{1}{n}$ and $\sin\phi = \frac{2}{n}$;

then the values of $\tan\theta$ are $\cot\left(\frac{\phi}{2}\right)$ and $\tan\left(\frac{\phi}{2}\right)$.

$$\text{Here } n\tan\theta = \frac{1}{\cos^2\theta} \quad (28)$$

$$= \sec^2\theta$$

$$= 1 + \tan^2\theta;$$

$$\therefore \tan^2\theta - n\tan\theta = -1;$$

$$\therefore \tan^2\theta - n\tan\theta + \frac{n^2}{4} = \frac{n^2}{4} - 1$$

$$= \frac{n^2 - 4}{4};$$

$$\therefore \tan\theta - \frac{n}{2} = \frac{\pm\sqrt{n^2 - 4}}{2};$$

$$\therefore \tan\theta = \frac{n \pm \sqrt{n^2 - 4}}{2}.$$

$$\text{Now } n = \frac{2}{\sin\phi} = 2\operatorname{cosec}\phi;$$

$$\therefore\ n^2 = 4\operatorname{cosec}^2\phi \text{ and } n^2 - 4 = 4\operatorname{cosec}^2\phi - 4 = 4(\operatorname{cosec}^2\phi - 1) = 4\cot^2\phi\,;$$

$$\therefore\ \sqrt{(n^2-4)} = 2\cot\phi\,;$$

$$\therefore\ \tan\theta = \frac{n \pm \sqrt{(n^2-4)}}{2}$$

$$= \operatorname{cosec}\phi \pm \cot\phi$$

$$= \cot\frac{\phi}{2} \text{ and } \tan\frac{\phi}{2}\quad (28).$$

144. One of the values of (θ) in the equation $2\sin^2\theta = \cos^3\theta$ is

$$\sec^{-1}\left(\sqrt[3]{\frac{1}{4} + \frac{1}{12}\sqrt{\frac{11}{3}}} + \sqrt[3]{\frac{1}{4} - \frac{1}{12}\sqrt{\frac{11}{3}}}\right).$$

Let $\sec\theta = z$.

$$\text{Then } \frac{1}{z^3} = 2\sin^2\theta = \frac{2\tan^2\theta}{\sec^2\theta}\quad (25)$$

$$= \frac{2(z^2-1)}{z^2}\,;$$

$$\therefore\ \frac{1}{z} = 2\,.\,(z^2-1)\,;$$

$$\therefore\ z^3 - z - \frac{1}{2} = 0.$$

$$\text{Let } z = \sqrt[3]{x} + \sqrt[3]{y}.$$

$$\text{Then } z^3 = x + y + 3\sqrt[3]{xy}\,.\,(\sqrt[3]{x} + \sqrt[3]{y})$$

$$= x + y + 3\sqrt[3]{xy}\,.\,z\,;$$

$$\therefore\ z^3 - 3\sqrt[3]{xy}\,.\,z - (x+y) = 0 = z^3 - z - \frac{1}{2}\,;$$

therefore by equating the homologous terms, we obtain

$$3\sqrt[3]{xy} = 1, \text{ and } x + y = \frac{1}{2}\,;$$

$$\therefore\ 27xy = 1, \text{ and } 4xy = \frac{4}{27}.$$

$$\text{Also } x^2+2xy+y^2=\frac{1}{4};$$

$$\therefore\ x^2-2xy+y^2=\frac{1}{4}-\frac{4}{27}=\frac{11}{3\times 36};$$

$$\therefore\ x-y=\frac{1}{6}\sqrt{\left(\frac{11}{3}\right)},$$

$$x+y=\frac{1}{2};$$

$$\therefore\ 2x=\frac{1}{2}+\frac{1}{6}\sqrt{\left(\frac{11}{3}\right)},$$

$$x=\frac{1}{4}+\frac{1}{12}\sqrt{\frac{11}{3}},$$

$$\text{and } y=\frac{1}{4}-\frac{1}{12}\sqrt{\frac{11}{3}};$$

$$\therefore\ \sqrt[3]{x}=\sqrt[3]{\frac{1}{4}+\frac{1}{12}\sqrt{\frac{11}{3}}},$$

$$\sqrt[3]{y}=\sqrt[3]{\frac{1}{4}-\frac{1}{12}\sqrt{\frac{11}{3}}};$$

$$\therefore\ z=\sqrt[3]{x}+\sqrt[3]{y}=\sqrt[3]{\frac{1}{4}+\frac{1}{12}\sqrt{\frac{11}{3}}}+\sqrt[3]{\frac{1}{4}-\frac{1}{12}\sqrt{\frac{11}{3}}};$$

$$\therefore\ \theta=\sec^{-1}\left(\sqrt[3]{\frac{1}{4}+\frac{1}{12}\sqrt{\frac{11}{3}}}+\sqrt[3]{\frac{1}{4}-\frac{1}{12}\sqrt{\frac{11}{3}}}\right).$$

CHAPTER VI.

SERIES.

145. **Sum** $\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}+\sin\frac{5\theta}{2}+\ldots\ldots\ldots$ to (n) terms,

$$\cos 0-\cos\theta=2\sin\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\quad(11)\text{ and }(32),$$

$$\cos\theta-\cos 2\theta=2\sin\frac{3\theta}{2}\cdot\sin\frac{\theta}{2}\quad(40),$$

$$\cos 2\theta-\cos 3\theta=2\sin\frac{5\theta}{2}\cdot\sin\frac{\theta}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\cos(n-1)\cdot\theta-\cos n\theta=2\sin(2n-1)\frac{\theta}{2}\cdot\sin\frac{\theta}{2};$$

$$\therefore\ \cos 0-\cos n\theta=2\sin\frac{\theta}{2}\cdot S_1;$$

$$\therefore\ S_1=\frac{1-\cos n\theta}{2\sin\frac{\theta}{2}}=\frac{2\sin^2\frac{n\theta}{2}}{2\sin\frac{\theta}{2}}=\frac{\sin^2\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

146. If $S_1=\dfrac{\cot^2\frac{\theta}{2}-1}{\operatorname{cosec}\theta\cdot\cot\frac{\theta}{2}}+\dfrac{\cot^2\frac{3\theta}{2}-1}{\operatorname{cosec}3\theta\cdot\cot\frac{3\theta}{2}}+\ldots\ldots\ldots$

$$\frac{\cot^2\left(\frac{2n-1}{2}\right)\theta-1}{\operatorname{cosec}(2n-1)\theta\cdot\cot\left(\frac{2n-1}{2}\right)\theta},$$

and $$S_2=\frac{\cot^2\frac{\theta}{2}+1}{\operatorname{cosec}\theta\,.\,\cot\frac{\theta}{2}}+\frac{\cot^2\frac{3\theta}{2}+1}{\operatorname{cosec}3\theta\,.\,\cot\frac{3\theta}{2}}+\dots\dots\dots$$

$$\frac{\cot^2\left(\frac{2n-1}{2}\right)\theta+1}{\operatorname{cosec}(2n-1)\theta\,.\,\cot\left(\frac{2n-1}{2}\right)\theta}.$$

Then $\dfrac{S_1}{S_2}=\dfrac{\sin 2n\theta}{2n\sin\theta}$,

$$\frac{\cot^2\frac{\theta}{2}-1}{\operatorname{cosec}\theta\,.\,\cot\frac{\theta}{2}}=\frac{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}{\operatorname{cosec}\theta}\ (23)\ =\frac{2\cot\theta}{\operatorname{cosec}\theta}\ (27)\ =2\cos\theta\quad(25),$$

$$\frac{\cot^2\frac{3\theta}{2}-1}{\operatorname{cosec}3\theta\,.\,\cot\frac{3\theta}{2}}=2\cos 3\theta,$$

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$$\frac{\cot^2\left(\frac{2n-1}{2}\right).\theta-1}{\operatorname{cosec}(2n-1)\theta\,.\,\cot\left(\frac{2n-1}{2}\right)\theta}=2\cos(2n-1)\theta;$$

$$\therefore\ S_1=2\{\cos\theta+\cos 3\theta+\cos 5\theta+\dots\dots\dots\cos(2n-1)\theta\}$$

$$=\frac{2\sin n\theta\,.\,\cos n\theta}{\sin\theta}\ (66)\ =\frac{\sin 2n\theta}{\sin\theta}.$$

Similarly $S_2=2+2+2+2+2+\dots\dots\dots$ to (n) terms $=2n$;

$$\therefore\ \frac{S_1}{S_2}=\frac{\sin 2n\theta}{2n\sin\theta}.$$

147. Sum $\dfrac{\sec(45+\theta)}{\sec 45\,.\,\sec 2\theta}+\dfrac{\sec(45+2\theta)}{\sec 45\,.\,\sec 4\theta}+\dots\dots\dots\dfrac{\sec(45+n\theta)}{\sec 45\,.\,\sec 2n\theta}$ in functions of $\dfrac{\theta}{2}$ and $\dfrac{n\theta}{2}$.

$$\frac{\sec(45+\theta)}{\sec 45\,.\,\sec 2\theta}=\cos\theta+\sin\theta.\ \text{(Prop. 18)},$$

$$\frac{\sec(45+2\theta)}{\sec 45 \,.\, \sec 4\theta} = \cos 2\theta + \sin 2\theta,$$

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$$\frac{\sec(45+n\theta)}{\sec 45 \,.\, \sec 2n\theta} = \cos n\theta + \sin n\theta;$$

$$\therefore\ S_1 = (\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots\ldots \cos n\theta)$$
$$+ (\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots\ldots \sin n\theta)$$

$$= \frac{\cos\left(\frac{n+1}{2}\right)\theta \,.\, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} + \frac{\sin\left(\frac{n+1}{2}\right)\theta \,.\, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \quad (67) \text{ and } (68)$$

$$= \left(\frac{\cos\frac{n\theta}{2} \,.\, \cos\frac{\theta}{2} - \sin\frac{n\theta}{2} \,.\, \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right.$$
$$\left. + \frac{\sin\frac{n\theta}{2} \,.\, \cos\frac{\theta}{2} + \cos\frac{n\theta}{2} \,.\, \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) . \sin\frac{n\theta}{2} \quad (36) \text{ and } (37)$$

$$= \left(\cos\frac{n\theta}{2} \,.\, \cot\frac{\theta}{2} - \sin\frac{n\theta}{2} + \sin\frac{n\theta}{2} \,.\, \cot\frac{\theta}{2} + \cos\frac{n\theta}{2}\right) . \sin\frac{n\theta}{2}$$

$$= \left\{\left(1 + \cot\frac{\theta}{2}\right) . \cos\frac{n\theta}{2} - \left(1 - \cot\frac{\theta}{2}\right) . \sin\frac{n\theta}{2}\right\} . \sin\frac{n\theta}{2}.$$

148. Sum $\tan\theta \,.\, \sec 2\theta + \tan 2\theta \,.\, \sec 4\theta + \tan 4\theta \,.\, \sec 8\theta + \ldots\ldots\ldots$ to (n) terms,

$$\cos 2\theta = \frac{\tan\theta}{\tan 2\theta - \tan\theta}; \quad (\text{Prop. } 30)$$

$$\therefore\ \tan\theta \,.\, \sec 2\theta = \tan 2\theta - \tan\theta \quad (23),$$

and similarly $\tan 2\theta \,.\, \sec 4\theta = \tan 4\theta - \tan 2\theta,$

and $\tan 4\theta \,.\, \sec 8\theta = \tan 8\theta - \tan 4\theta,$

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$$\tan 2^{n-1}\theta \,.\, \sec 2^n\theta = \tan 2^n\theta - \tan 2^{n-1}\theta;$$

$$\therefore\ S_1 = \tan 2^n\theta - \tan\theta.$$

149. Sum $\tan\frac{\theta}{2} + \tan\theta + \tan 2\theta + \ldots\ldots \tan 2^{n-2}\theta + \cot\theta$

$+ \cot 2\theta + \cot 4\theta + \ldots\ldots \cot 2^{n-1}\theta,$

$$2\cot\theta = \cot\frac{\theta}{2} - \tan\frac{\theta}{2} \quad (28);$$

$$\therefore \tan\frac{\theta}{2} + \cot\theta = \cot\frac{\theta}{2} - \cot\theta;$$

and similarly $\tan\theta + \cot 2\theta = \cot\theta - \cot 2\theta,$

and $\tan 2\theta + \cot 4\theta = \cot 2\theta - \cot 4\theta,$

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$$\tan 2^{n-2}\theta + \cot 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta;$$

$$\therefore S_1 = \cot\frac{\theta}{2} - \cot 2^{n-1}\theta.$$

150. Sum $\text{chd}\,2\theta\,.\,\text{vers}\,\theta + \frac{1}{2}\,\text{chd}\,4\theta\,.\,\text{vers}\,2\theta$

$+ \frac{1}{4}\,\text{chd}\,8\theta\,.\,\text{vers}\,4\theta + \ldots\ldots$ to (n) terms.

$\text{chd}\,2\theta\,.\,\text{vers}\,\theta = (1-\cos\theta)\,2\sin\theta = 2\sin\theta - \sin 2\theta,$

$\frac{1}{2}\,\text{chd}\,4\theta\,.\,\text{vers}\,2\theta = (1-\cos 2\theta)\,.\,\sin 2\theta = \sin 2\theta - \frac{1}{2}\sin 4\theta,$

$\frac{1}{4}\,\text{chd}\,8\theta\,.\,\text{vers}\,4\theta = (1-\cos 4\theta)\,\frac{1}{2}\sin 4\theta = \frac{1}{2}\sin 4\theta - \frac{1}{4}\sin 8\theta,$

$\frac{1}{8}\,\text{chd}\,16\theta\,.\,\text{vers}\,8\theta = (1-\cos 8\theta)\,\frac{1}{4}\sin 8\theta = \frac{1}{4}\sin 8\theta - \frac{1}{8}\sin 16\theta.$

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$$\frac{1}{2^{n-1}}\,\text{chd}\,2^n\theta\,.\,\text{vers}\,2^{n-1}\theta = (1-\cos 2^{n-1}\theta)\,\frac{1}{2^{n-2}}\,.\,\sin 2^{n-1}\theta$$

$$= \frac{1}{2^{n-2}}\sin 2^{n-1}\theta - \frac{1}{2^{n-1}}\,.\,\sin 2^n\theta;$$

$$\therefore S_1 = 2\sin\theta - \frac{1}{2^{n-1}}\,.\,\sin 2^n\theta$$

$$= 2\,(\sin\theta - \frac{1}{2^n}\,.\,\sin 2^n\theta).$$

151. Sum $\dfrac{\sin\left(30+\dfrac{\theta}{2}\right).\sin\left(30-\dfrac{\theta}{2}\right)}{\sin 2\theta}+\dfrac{\sin(30+\theta).\sin(30-\theta)}{\sin 2^2\theta}$

$+ \ldots\ldots\ldots \dfrac{\sin(30+2^{n-2}\theta).\sin(30-2^{n-2}\theta)}{\sin 2^n\theta}$.

$$\frac{\sin\left(30+\dfrac{\theta}{2}\right).\sin\left(30-\dfrac{\theta}{2}\right)}{\sin 2\theta}=\tfrac{1}{4}\operatorname{cosec}\theta-\tfrac{1}{4}\operatorname{cosec}2\theta \quad \text{(Prop. 16)};$$

and similarly $\dfrac{\sin(30+\theta).\sin(30-\theta)}{\sin 2^2.\theta}=\tfrac{1}{4}\operatorname{cosec}2\theta-\tfrac{1}{4}\operatorname{cosec}4\theta,$

$$\frac{\sin(30+2\theta).\sin(30-2\theta)}{\sin 2^3.\theta}=\tfrac{1}{4}\operatorname{cosec}4\theta-\tfrac{1}{4}\operatorname{cosec}8\theta.$$

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$$\frac{\sin(30+2^{n-2}\theta).\sin(30-2^{n-2}\theta)}{\sin 2^n\theta}=\tfrac{1}{4}\operatorname{cosec}2^{n-1}\theta-\tfrac{1}{4}\operatorname{cosec}2^n\theta;$$

$$\therefore\ S_1=\tfrac{1}{4}\operatorname{cosec}\theta-\tfrac{1}{4}\operatorname{cosec}2^n\theta$$

$$=\tfrac{1}{4}(\operatorname{cosec}\theta-\operatorname{cosec}2^n\theta).$$

152. If $\dfrac{1+S_1}{1-S_1}=\tfrac{1}{2}\left(\cot^2\dfrac{\theta}{2}+\tan^2\dfrac{\theta}{2}\right)$.

Then $S_1=\cot^2\theta\,(1-\cot^2\theta+\cot^4\theta-\cot^6\theta+\ldots\ldots\ldots)$.

Here $\dfrac{2+2S_1}{1-S_1}=\cot^2\dfrac{\theta}{2}+\tan^2\dfrac{\theta}{2}$

$$=\left(\cot\frac{\theta}{2}+\tan\frac{\theta}{2}\right)^2-2\cot\frac{\theta}{2}.\tan\frac{\theta}{2}$$

$$=4\operatorname{cosec}^2\theta-2;$$

$$\therefore\ \frac{1+S_1}{1-S_1}=2\operatorname{cosec}^2\theta-1;$$

$$\therefore\ 2\operatorname{cosec}^2\theta = \frac{1+S_1}{1-S_1} + 1$$

$$= \frac{2}{1-S_1};$$

$$\therefore\ \operatorname{cosec}^2\theta = \frac{1}{1-S_1};$$

$$\therefore\ 1 - S_1 = \frac{1}{\operatorname{cosec}^2\theta}$$

$$= \sin^2\theta$$

$$= 1 - \cos^2\theta;$$

$$\therefore\ S_1 = \cos^2\theta = \frac{\cot^2\theta}{\operatorname{cosec}^2\theta} \quad (25)$$

$$= \cot^2\theta\left(\frac{1}{1+\cot^2\theta}\right)$$

$$= \cot^2\theta\,(1 - \cot^2\theta + \cot^4\theta - \cot^6\theta + \ldots\ldots).$$

153. If $\sqrt{S_1} = \dfrac{\sqrt{2}}{\sqrt{\cot\frac{\theta}{2}} - \sqrt{\tan\frac{\theta}{2}}}$ and $\sqrt{S_2} = \dfrac{\sqrt{2}}{\sqrt{\cot\frac{\theta}{2}} + \sqrt{\tan\frac{\theta}{2}}}$.

Then $S_1 : S_2 :: 1 + \sin\theta : 1 - \sin\theta$,

$$S_1 = \frac{2}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2} - 2\sqrt{\cot\frac{\theta}{2}\,.\,\tan\frac{\theta}{2}}} = \frac{2}{2\operatorname{cosec}\theta - 2} = \frac{1}{\operatorname{cosec}\theta - 1}$$

$$= \frac{\sin\theta}{1-\sin\theta},$$

$$S_2 = \frac{2}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2} + 2\sqrt{\cot\frac{\theta}{2}\,.\,\tan\frac{\theta}{2}}} = \frac{2}{2\operatorname{cosec}\theta + 2} = \frac{1}{\operatorname{cosec}\theta + 1}$$

$$= \frac{\sin\theta}{1+\sin\theta};$$

$$\therefore\ S_1 : S_2 :: \frac{\sin\theta}{1-\sin\theta} : \frac{\sin\theta}{1+\sin\theta}$$

$$:: 1 + \sin\theta : 1 - \sin\theta.$$

154. Sum $\dfrac{\text{chd}\,2\theta}{\text{chd}\,\theta}-\dfrac{\text{chd}\,4\theta}{\text{chd}\,2\theta}+\dfrac{\text{chd}\,6\theta}{\text{chd}\,3\theta}-\ldots\ldots\ldots\ldots\pm\dfrac{\text{chd}\,2n\theta}{\text{chd}\,n\theta}$.

Let $\theta=2\phi$.

$$\text{Then }\frac{\text{chd}\,2\theta}{\text{chd}\,\theta}=\frac{\text{chd}\,4\phi}{\text{chd}\,2\phi}=\frac{2\sin 2\phi}{2\sin\phi}=2\cos\phi.$$

$$\text{And now let }2\cos\phi=x+\frac{1}{x}.$$

Then $2\cos n\phi=x^n+\dfrac{1}{x^n}$, where ($n$) may be any number, integral or fractional (57);

$$\therefore\ \frac{\text{chd}\,2\theta}{\text{chd}\,\theta}=x+\frac{1}{x},$$

$$\text{and similarly }\frac{\text{chd}\,4\theta}{\text{chd}\,2\theta}=x^2+\frac{1}{x^2},$$

$$\text{and }\frac{\text{chd}\,6\theta}{\text{chd}\,3\theta}=x^3+\frac{1}{x^3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\text{chd}\,2n\theta}{\text{chd}\,n\theta}=x^n+\frac{1}{x^n};$$

$$\therefore\ S_1=(x-x^2+x^3-x^4+\ldots\ldots\pm x^n)+\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\ldots\ldots\pm\frac{1}{x^n}\right)$$

$$=\frac{x\pm x^{n+1}}{x+1}+\frac{\dfrac{1}{x}\pm\dfrac{1}{x^{n+1}}}{\dfrac{1}{x}+1}$$

$$=\frac{x+1\pm\left(x^{n+1}+\dfrac{1}{x^n}\right)}{x+1}$$

$$=\frac{\left(x^{\frac{1}{2}}+\dfrac{1}{x^{\frac{1}{2}}}\right)\pm\left(x^{n+\frac{1}{2}}+\dfrac{1}{x^{n+\frac{1}{2}}}\right)}{x^{\frac{1}{2}}+\dfrac{1}{x^{\frac{1}{2}}}}$$

$$= \frac{\cos\frac{\phi}{2} \pm \cos\left(\frac{2n+1}{2}\right)\phi}{\cos\frac{\phi}{2}} \qquad (57)$$

$$= \frac{\cos\frac{\theta}{4} \pm \cos\left(\frac{2n+1}{4}\right)\theta}{\cos\frac{\theta}{4}}$$

$$= \frac{2\cos\left(\frac{n+1}{4}\right)\theta \,.\, \cos\frac{n}{4}\,.\,\theta}{\cos\frac{\theta}{4}} \text{ or } \frac{2\sin\left(\frac{n+1}{4}\right)\theta \,.\, \sin\frac{n}{4}\,.\,\theta}{\cos\frac{\theta}{4}} \qquad (40)$$

according as (n) is *odd* or *even*.

155. $$\frac{\cos(45-\theta)+\cos(45-4\theta)+\cos(45-7\theta)+\ldots\text{to } (n) \text{ terms}}{\sin(45-\theta)+\sin(45-4\theta)+\sin(45-7\theta)+\ldots\text{to } (n) \text{ terms}}$$

$$= \tan\left\{45+\left(\frac{3n-1}{2}\right)\theta\right\}. \quad \text{Required proof.}$$

$$\cos(45-\theta) = \cos 45\,.\,\cos\theta + \sin 45\,.\,\sin\theta = \cos 45\,.\,(\cos\theta+\sin\theta)$$

(37) and (15),

and similarly $\cos(45-4\theta) = \cos 45\,.\,(\cos 4\theta + \sin 4\theta)$,

and $\cos(45-7\theta) = \cos 45\,.\,(\cos 7\theta + \sin 7\theta)$

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$$\cos\{45-(3n-2)\theta\} = \cos 45\,.\,\{\cos(3n-2)\theta + \sin(3n-2)\theta\};$$

$$\therefore \cos(45-\theta)+\cos(45-4\theta)+\cos(45-7\theta)+\ldots\cos\{45-(3n-2)\theta\}$$

$$= \cos 45\,.\,\{\cos\theta + \cos 4\theta + \ldots\cos(3n-2)\theta\}$$

$$+ \cos 45\,.\,\{\sin\theta + \sin 4\theta + \ldots\sin(3n-2)\theta\}$$

$$= \cos 45\,.\left\{\frac{\cos\left(\frac{3n-1}{2}\right)\theta\,.\,\sin\frac{3n}{2}\,.\,\theta}{\sin\frac{3\theta}{2}}\right\} + \cos 45\,.\left\{\frac{\sin\left(\frac{3n-1}{2}\right)\theta\,.\,\sin\frac{3n}{2}\,.\,\theta}{\sin\frac{3\theta}{2}}\right\}$$

(69) and (70).

In the same manner it may be shewn that

$$\sin(45-\theta)+\sin(45-4\theta)+\sin(45-7\theta)+\dots\sin\{45-(3n-2)\,\theta\}$$

$$=\cos 45.\left\{\frac{\cos\left(\frac{3n-1}{2}\right)\theta.\sin\frac{3n}{2}.\theta}{\sin\frac{3\theta}{2}}\right\}-\cos 45\left\{\frac{\sin\left(\frac{3n-1}{2}\right)\theta.\sin\left(\frac{3n}{2}\right)\theta}{\sin\frac{3\theta}{2}}\right\};$$

$$\therefore \frac{\cos(45-\theta)+\cos(45-4\theta)+\cos(45-7\theta)+\dots\cos\{45-(3n-2)\,\theta\}}{\sin(45-\theta)+\sin(45-4\theta)+\sin(45-7\theta)+\dots\sin\{45-(3n-2)\,\theta\}}$$

$$=\frac{\cos\left(\frac{3n-1}{2}\right)\theta+\sin\left(\frac{3n-1}{2}\right)\theta}{\cos\left(\frac{3n-1}{2}\right)\theta-\sin\left(\frac{3n-1}{2}\right)\theta}$$

$$=\frac{1+\dfrac{\sin\left(\frac{3n-1}{2}\right)\theta}{\cos\left(\frac{3n-1}{2}\right)\theta}}{1-\dfrac{\sin\left(\frac{3n-1}{2}\right)\theta}{\cos\left(\frac{3n-1}{2}\right)\theta}}$$

$$=\frac{1+\tan\left(\frac{3n-1}{2}\right)\theta}{1-\tan\left(\frac{3n-1}{2}\right)\theta}$$

$$=\tan\left\{45+\left(\frac{3n-1}{2}\right)\theta\right\}\quad (12)\text{ and }(38).$$

156. Sum $\frac{3}{1.5}-\frac{5}{3.7}+\frac{11}{9.13}-\frac{13}{11.15}+\frac{19}{17.21}-\dots\dots$

$$\frac{3}{1.5}-\frac{5}{3.7}+\frac{11}{9.13}-\dots\dots=\frac{1}{2}\left(\frac{6}{1.5}-\frac{10}{3.7}+\frac{22}{9.13}-\dots\dots\right)$$

$$=\frac{1}{2}\left(\frac{5+1}{1.5}-\frac{7+3}{7.3}+\frac{9+13}{9.13}-\dots\dots\right)$$

$$=\frac{1}{2}\left\{1+\frac{1}{5}-\left(\frac{1}{3}+\frac{1}{7}\right)+\left(\frac{1}{9}+\frac{1}{13}\right)-\dots\dots\right\}$$

$$=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\,\ldots\ldots\ldots\right)$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)\quad (59) \text{ and } (12)$$

$$=\frac{\pi}{8}.$$

157. Sum the series:

$$3^{-\frac{1}{2}}.\left(\frac{3+1}{1}\right)-3^{-\frac{3}{2}}.\left(\frac{3^3+1}{3}\right)+3^{-\frac{5}{2}}.\left(\frac{3^5+1}{5}\right)-3^{-\frac{7}{2}}.\left(\frac{3^7+1}{7}\right)+\,\ldots\ldots\ldots$$

$$3^{-\frac{1}{2}}.\left(\frac{3+1}{1}\right)-3^{-\frac{3}{2}}.\left(\frac{3^3+1}{3}\right)+3^{-\frac{5}{2}}.\left(\frac{3^5+1}{5}\right)-3^{-\frac{7}{2}}.\left(\frac{3^7+1}{7}\right)+\,\ldots\ldots\ldots$$

$$=\frac{1}{3^{\frac{1}{2}}}.\left(\frac{3+1}{1}\right)-\frac{1}{3^{\frac{3}{2}}}.\left(\frac{3^3+1}{3}\right)+\frac{1}{3^{\frac{5}{2}}}.\left(\frac{3^5+1}{5}\right)-\,\ldots\ldots\ldots$$

$$=\left(3^{\frac{1}{2}}+\frac{1}{3^{\frac{1}{2}}}\right)-\frac{1}{3}.\left(3^{\frac{3}{2}}+\frac{1}{3^{\frac{3}{2}}}\right)+\frac{1}{5}.\left(3^{\frac{5}{2}}+\frac{1}{3^{\frac{5}{2}}}\right)-\,\ldots\ldots\ldots$$

$$=\left(\tan\frac{\pi}{3}+\tan\frac{\pi}{6}\right)-\frac{1}{3}.\left(\tan^3\frac{\pi}{3}+\tan^3\frac{\pi}{6}\right)+\frac{1}{5}.\left(\tan^5\frac{\pi}{3}+\tan^5\frac{\pi}{6}\right)-\ldots\ldots\ldots$$

$$=(\tan\frac{\pi}{3}-\frac{1}{3}.\tan^3\frac{\pi}{3}+\,\ldots\ldots\ldots)+(\tan\frac{\pi}{6}-\frac{1}{3}.\tan^3\frac{\pi}{6}+\,\ldots\ldots\ldots)$$

$$=\frac{\pi}{3}+\frac{\pi}{6}\quad (59)$$

$$=\frac{\pi}{2}.$$

158. If $S_1=1-\frac{1}{3}.3+\frac{1}{5}.3^2-\,\ldots\ldots\ldots$

and $S_2=1-\frac{1}{3}.3^{-1}+\frac{1}{5}.3^{-2}-\,\ldots\ldots\ldots$

Then $S_1 : S_2 :: 2 : 3$,

$$S_1=\frac{1}{\sqrt{3}}.(3^{\frac{1}{2}}-\frac{1}{3}.3^{\frac{3}{2}}+\frac{1}{5}.3^{\frac{5}{2}}-\,\ldots\ldots\ldots)$$

$$=\frac{1}{\sqrt{3}}.(\tan\frac{\pi}{3}-\frac{1}{3}.\tan^3\frac{\pi}{3}+\frac{1}{5}.\tan^5\frac{\pi}{3}-\,\ldots\ldots\ldots)\quad (19)$$

$$=\frac{1}{\sqrt{3}}.\left(\frac{\pi}{3}\right)\quad (59).$$

$$\text{Again, } S_2 = \sqrt{3} \cdot \left(\frac{1}{3^{\frac{1}{2}}} - \frac{1}{3} \cdot \frac{1}{3^{\frac{3}{2}}} + \frac{1}{5} \cdot \frac{1}{3^{\frac{5}{2}}} - \ldots\ldots\ldots\right)$$

$$= \sqrt{3} \cdot \left(\tan\frac{\pi}{6} - \frac{1}{3} \cdot \tan^3\frac{\pi}{6} + \frac{1}{5} \cdot \tan^5\frac{\pi}{6} - \ldots\ldots\ldots\right) \quad (20)$$

$$= \sqrt{3} \cdot \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\pi}{3};$$

$$\therefore\ S_1 : S_2 :: \frac{1}{\sqrt{3}} : \frac{\sqrt{3}}{2}$$

$$:: 2 : 3.$$

159. Sum the series:

$$\tan(\theta - 45) + \frac{1}{3}\tan^3(\theta - 45) + \frac{1}{5}\tan^5(\theta - 45) + \ldots\ldots$$

$$\text{Let } \tan\theta = \frac{1+z}{1-z}.$$

$$\text{Then } \tan(\theta - 45) = \frac{\tan\theta - \tan 45}{1 + \tan\theta \cdot \tan 45} = \frac{\frac{1+z}{1-z} - 1}{1 + \frac{1+z}{1-z}} = z;$$

$$\therefore\ S_1 = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \ldots\ldots$$

$$\text{Now } \log_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots$$

$$\text{and } \log_e(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots\ldots (75);$$

$$\therefore\ \log_e(1+z) - \log_e(1-z) = 2 \cdot \left(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots\right);$$

$$\therefore\ S_1 = \tfrac{1}{2}\{\log_e(1+z) - \log_e(1-z)\}$$

$$= \tfrac{1}{2}\log_e\left(\frac{1+z}{1-z}\right) \quad (72)$$

$$= \log_e \sqrt{\left(\frac{1+z}{1-z}\right)} \quad (71)$$

$$= \log_e \sqrt{(\tan\theta)}.$$

160. Sum the series:

$$\frac{1}{x}\left(\frac{x^3-1}{1}\right)-\frac{1}{x^2}\left(\frac{x^5-1}{2}\right)+\frac{1}{x^3}\cdot\left(\frac{x^7-1}{3}\right)-\frac{1}{x^4}\cdot\left(\frac{x^9-1}{4}\right)+\ldots\ldots$$

$$S_1=x^2-\frac{1}{x}-\frac{1}{2}\left(x^3-\frac{1}{x^2}\right)+\frac{1}{3}\left(x^4-\frac{1}{x^3}\right)-\frac{1}{4}\left(x^5-\frac{1}{x^4}\right)+\ldots$$

$$=x\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots\right)-\left(\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{3}\cdot\frac{1}{x^3}-\frac{1}{4}\cdot\frac{1}{x^4}+\ldots\right)$$

$$=x\,.\log_e(1+x)-\log_e\left(1+\frac{1}{x}\right)\quad(75)$$

$$=x\,.\log_e(1+x)-\log_e\left(\frac{1+x}{x}\right)$$

$$=x\,.\log_e(1+x)-\log_e(1+x)+\log_e x\quad(72)$$

$$=(x-1)\log_e(1+x)+\log_e x.$$

161. Sum $\frac{1}{x^2}+\frac{1}{2}\cdot\left(\frac{1}{x^2}\right)^2+\frac{1}{3}\cdot\left(\frac{1}{x^2}\right)^3+\frac{1}{4}\cdot\left(\frac{1}{x^2}\right)^4+\ldots\ldots$

and apply it to shew that

$$\frac{1}{2}\cos^2\theta+\frac{1}{4}\cos^4\theta+\frac{1}{6}\cos^6\theta+\ldots\ldots=\log_e\operatorname{cosec}\theta.$$

Let $\frac{1}{x^2}=z$.

Then $S_1=z+\frac{1}{2}z^2+\frac{1}{3}z^3+\frac{1}{4}z^4+\ldots\ldots$

$$=-\left(-z-\frac{1}{2}z^2-\frac{1}{3}z^3-\frac{1}{4}z^4-\ldots\ldots\right)$$

$$=-\log_e(1-z)\quad(75)$$

$$=\log_e 1-\log_e(1-z)$$

$$=\log_e\left(\frac{1}{1-z}\right)\quad(72)$$

$$=\log_e\left(\frac{x^2}{x^2-1}\right),$$

$$\frac{1}{2}\cos^2\theta + \frac{1}{4}\cos^4\theta + \frac{1}{6}\cos^6\theta + \ldots\ldots$$

$$= \frac{1}{2} \cdot \left\{\left(\frac{1}{\sec^2\theta}\right) + \frac{1}{2} \cdot \left(\frac{1}{\sec^4\theta}\right) + \frac{1}{3}\left(\frac{1}{\sec^6\theta}\right) + \ldots\ldots\right\}$$

$$= \frac{1}{2}\left\{\log_\epsilon\left(\frac{\sec^2\theta}{\tan^2\theta}\right)\right\}$$

$$= \log_\epsilon\left(\frac{\sec\theta}{\tan\theta}\right) \quad (71)$$

$$= \log_\epsilon\left(\frac{1}{\sin\theta}\right) \quad (25)$$

$$= \log_\epsilon\,(\text{cosec}\,\theta) \quad (23).$$

162. Sum $\tan\theta + \frac{1}{3} \cdot \tan^3\theta + \frac{1}{5} \cdot \tan^5\theta + \frac{1}{7} \cdot \tan^7\theta + \ldots\ldots\ldots$

Let $\tan\theta = x$.

$$\text{Then } S_1 = x + \frac{1}{3} \cdot x^3 + \frac{1}{5} \cdot x^5 + \frac{1}{7} \cdot x^7 + \ldots\ldots\ldots$$

$$= \log_\epsilon \sqrt{\left(\frac{1+x}{1-x}\right)} \quad \text{(Prop. 159.)}$$

$$= \log_\epsilon \sqrt{\left(\frac{1+\tan\theta}{1-\tan\theta}\right)}$$

$$= \log_\epsilon \sqrt{\tan(45+\theta)} \quad (38) \text{ and } (12).$$

163. Sum $\frac{1}{2} \cdot \tan^2\theta + \frac{1}{4} \cdot \tan^4\theta + \frac{1}{6} \cdot \tan^6\theta + \ldots\ldots\ldots$

Let $\tan\theta = \frac{1}{x}$.

$$\text{Then } S_1 = \frac{1}{2} \cdot \left(\frac{1}{x^2} + \frac{1}{2} \cdot \frac{1}{x^4} + \frac{1}{3} \cdot \frac{1}{x^6} + \ldots\ldots\ldots\right)$$

$$= \frac{1}{2} \cdot \log_\epsilon\left(\frac{x^2}{x^2-1}\right) \quad \text{(Prop. 161.)}$$

$$= \log_\epsilon \sqrt{\left(\frac{x^2}{x^2-1}\right)}$$

$$= \log_\epsilon \sqrt{\left(\frac{1}{1-\frac{1}{x^2}}\right)}$$

$$= \log_\epsilon \sqrt{\left(\frac{1}{1-\tan^2\theta}\right)}.$$

$$= \log_\epsilon \cos\theta - \frac{1}{2}\log_\epsilon \cos 2\theta.$$

164. Sum $\tan\theta + \frac{1}{5} \cdot \tan^5\theta + \frac{1}{9} \cdot \tan^9\theta + \ldots\ldots\ldots$

Let $\tan\theta = x$.

Then $S_1 = x + \frac{1}{5} x^5 + \frac{1}{9} x^9 + \ldots\ldots\ldots$

Now

$$\log_\epsilon \sqrt{\tan(45+\theta)} = x + \frac{1}{3} \cdot x^3 + \frac{1}{5} \cdot x^5 + \frac{1}{7} \cdot x^7 + \frac{1}{9} \cdot x^9 + \ldots\ldots\ldots \quad \text{(Prop. 162)},$$

$$\text{and } \theta = x - \frac{1}{3} \cdot x^3 + \frac{1}{5} \cdot x^5 - \frac{1}{7} \cdot x^7 + \frac{1}{9} \cdot x^9 - \ldots\ldots\ldots \quad (59);$$

$$\therefore \log_\epsilon \sqrt{\tan(45+\theta)} + \theta = 2 \cdot (x + \frac{1}{5} \cdot x^5 + \frac{1}{9} \cdot x^9 + \ldots\ldots\ldots);$$

$$\therefore S_1 = \frac{1}{2} \cdot \{\log_\epsilon \sqrt{\tan(45+\theta)} + \theta\}.$$

165. Sum $1 + \frac{x^2}{1.3.4} + \frac{x^4}{1.3.4.5.6} + \frac{x^6}{1.3.4.5.6.7.8} + \ldots\ldots\ldots$

$$S_1 = \frac{2}{x^2} \cdot \left(\frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \ldots\ldots\ldots\right)$$

$$= \frac{2}{x^2} \cdot (1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \ldots\ldots\ldots) - \frac{2}{x^2}.$$

$$\text{Now } \epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots\ldots\ldots$$

$$\text{and } \epsilon^{-x} = 1 - x + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \ldots\ldots\ldots \quad (74);$$

$$\therefore \epsilon^x + \epsilon^{-x} = 2\,(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \ldots\ldots\ldots);$$

$$\therefore S_1 = \frac{2}{x^2} \cdot \left(\frac{\epsilon^x + \epsilon^{-x}}{2}\right) - \frac{2}{x^2}$$

$$= \frac{1}{x^2} \cdot (\epsilon^x + \epsilon^{-x} - 2)$$

$$= \left(\frac{\epsilon^{\sqrt{x}} - \epsilon^{-\sqrt{x}}}{x}\right)^2.$$

166. $$\frac{\epsilon^{x} + \frac{1}{3}\cdot\epsilon^{3x} + \frac{1}{5}\cdot\epsilon^{5x} + \frac{1}{7}\cdot\epsilon^{7x} + \ldots\ldots\ldots}{\frac{1}{2}\cdot\epsilon^{2x} + \frac{1}{4}\cdot\epsilon^{4x} + \frac{1}{6}\cdot\epsilon^{6x} + \ldots\ldots\ldots}$$

$$= \frac{\log_{\epsilon}(1-\epsilon^{x}) - \log_{\epsilon}(1+\epsilon^{x})}{\log_{\epsilon}(1-\epsilon^{2x})}.$$ Required proof.

$$\log_{\epsilon}(1-\epsilon^{x}) = -\epsilon^{x} - \frac{1}{2}\cdot\epsilon^{2x} - \frac{1}{3}\cdot\epsilon^{3x} - \frac{1}{4}\cdot\epsilon^{4x} - \ldots\ldots\ldots),$$

$$\log_{\epsilon}(1+\epsilon^{x}) = \epsilon^{x} - \frac{1}{2}\cdot\epsilon^{2x} + \frac{1}{3}\cdot\epsilon^{3x} - \frac{1}{4}\cdot\epsilon^{4x} + \ldots\ldots\ldots \quad (75);$$

$$\therefore \log_{\epsilon}(1-\epsilon^{x}) - \log_{\epsilon}(1+\epsilon^{x}) = -2(\epsilon^{x} + \frac{1}{3}\cdot\epsilon^{3x} + \frac{1}{5}\cdot\epsilon^{5x} + \ldots\ldots\ldots$$

and $$\log_{\epsilon}(1-\epsilon^{x}) + \log_{\epsilon}(1+\epsilon^{x}) = -2\left(\frac{1}{2}\cdot\epsilon^{2x} + \frac{1}{4}\cdot\epsilon^{4}x + \frac{1}{6}\cdot\epsilon^{6x} + \ldots\ldots\ldots\right);$$

$$\therefore \frac{\epsilon^{x} + \frac{1}{3}\cdot\epsilon^{3x} + \frac{1}{5}\cdot\epsilon^{5x} + \ldots\ldots\ldots}{\frac{1}{2}\cdot\epsilon^{2x} + \frac{1}{4}\cdot\epsilon^{4x} + \frac{1}{6}\cdot\epsilon^{6x} + \ldots\ldots\ldots} = \frac{\log_{\epsilon}(1-\epsilon^{x}) - \log_{\epsilon}(1+\epsilon^{x})}{\log_{\epsilon}(1-\epsilon^{x}) + \log_{\epsilon}(1+\epsilon^{x})}$$

$$= \frac{\log_{\epsilon}(1-\epsilon^{x}) - \log_{\epsilon}(1+\epsilon^{x})}{\log\{(1+\epsilon^{x})\cdot(1-\epsilon^{x})\}} \quad (72)$$

$$= \frac{\log_{\epsilon}(1-\epsilon^{x}) - \log_{\epsilon}(1+\epsilon^{x})}{\log(1-\epsilon^{2x})}.$$

CHAPTER VII.

MISCELLANEOUS.

167. **Required** the arc whose supplement is to its complement as four to one.

Let θ be the arc.

$$\text{Then } \pi-\theta : \frac{\pi}{2}-\theta :: 4 : 1;$$

$$\therefore\ \pi-\theta=2\pi-4\theta;\quad \therefore\ 3\theta=\pi \text{ and } \theta=\frac{\pi}{3}.$$

168. Required the arc the sum of whose supplement and complement is to their difference as two to one.

Let θ be the arc.

$$\text{Then } (\pi-\theta)+\left(\frac{\pi}{2}-\theta\right) : (\pi-\theta)-\left(\frac{\pi}{2}-\theta\right) :: 2 : 1;$$

$$\therefore\ \frac{3\pi}{2}-2\theta : \frac{\pi}{2} :: 2 : 1;$$

$$\therefore\ \frac{3\pi}{2}-2\theta=\pi;$$

$$\therefore\ 2\theta=\frac{\pi}{2} \text{ and } \theta=\frac{\pi}{4}.$$

169. If $\tan\frac{\theta}{2}=\frac{1}{2}$ and $\tan\frac{\phi}{2}=\frac{1}{3}$, then $\operatorname{vers}\theta=2\operatorname{vers}\phi$.

$$\operatorname{vers}\phi=1-\cos\phi=1-\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}=\frac{2\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\quad(30)$$

$$=\frac{\frac{2}{9}}{\frac{10}{9}}=\frac{2}{10}=\frac{1}{5},$$

$$\text{vers}\,\theta = 1 - \cos\theta = 1 - \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{2\tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

$$= \frac{\frac{2}{4}}{\frac{5}{4}} = \frac{2}{5}, \text{ and } \therefore\ = 2\,\text{vers}\,\phi.$$

170. $\cos\theta = \text{chd}^2\theta$. Find the numerical values of $\sin\theta$, $\cos\theta$, $\text{vers}\,\theta$, $\text{chd}\,\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$.

$$\cos\theta = \text{chd}^2\theta = 2 - 2\cos\theta \quad (24); \quad \therefore\ 3\cos\theta = 2 \text{ and } \cos\theta = \frac{2}{3};$$

$$\therefore\ \text{chd}\,\theta = \sqrt{\left(\frac{2}{3}\right)},$$

$$\text{and vers}\,\theta = 1 - \cos\theta = 1 - \frac{2}{3} = \frac{1}{3},$$

$$\sin\theta = \sqrt{(1 - \cos^2\theta)} = \sqrt{\left(1 - \frac{4}{9}\right)} = \frac{\sqrt{5}}{3},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1\tfrac{1}{2},$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{5}}{3}} = \frac{3}{\sqrt{5}},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2},$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{2}{\sqrt{5}}.$$

171. Given $\sin 18 = \frac{\sqrt{5} - 1}{4}$ to find the numerical value of $\frac{\tan 36}{\tan 18}$.

$$\frac{\tan 36}{\tan 18} = \frac{2\tan 18}{(1 - \tan^2 18)\tan 18} \quad (33) \quad = \frac{2}{1 - \tan^2 18}$$

$$=\frac{2\cos^2 18}{\cos^2 18-\sin^2 18}=\frac{2-2\sin^2 18}{1-2\sin^2 18}.$$

$$\text{Now } \sin 18=\frac{\sqrt{5}-1}{4};$$

$$\therefore\ \sin^2 18=\frac{6-2\sqrt{5}}{16}=\frac{3-\sqrt{5}}{8};$$

$$\therefore\ 2\sin^2 18=\frac{3-\sqrt{5}}{4};$$

$$\therefore\ \frac{\tan 36}{\tan 18}=\frac{2-\dfrac{3-\sqrt{5}}{4}}{1-\dfrac{3-\sqrt{5}}{4}}=\frac{5+\sqrt{5}}{1+\sqrt{5}}=\sqrt{5}.$$

172. The number of the sides of one regular polygon is to that of another as 3 to 2. Also the sum of all the angles of the two polygons is to their difference as 11 to 3.

Required the nature of the polygons and the value of an angle of each.

Let (x) be the number of the sides of the *first* polygon,

(θ) one of its angles,

y be the number of the sides of the *second* polygon,

ϕ one of its angles.

Then since both the polygons are regular,

$$x\theta=\text{sum of angles of the first}=2x\cdot\frac{\pi}{2}-4\cdot\frac{\pi}{2}$$

(Euclid, I. 32, Cor. 1)

$$=(x-2)\pi.$$

Similarly $y\phi=(y-2)\pi$.

$$\text{Now } x:y::3:2;\quad \therefore\ 2x=3y,$$

$$\text{and } 11:3::x\theta+y\phi:x\theta-y\phi;$$

$$\therefore\ 14:8::2x\theta:2y\phi,$$

$$\text{and } 7:4::x\theta:y\phi$$

$$::x-2:y-2;$$

$$\therefore\ 7y - 14 = 4x - 8 = 6y - 8;$$

$\therefore\ y = 6$ and the 2nd polygon is a *Hexagon.*

$x = \frac{3y}{2} = 9$ and the 1st ...... *Nonagon.*

$$\text{Also } \theta = \left(\frac{x-2}{x}\right)\pi = 140°,$$

$$\phi = \left(\frac{y-2}{y}\right).\pi = 120°.$$

173. Required that arc which is divisible without remainder, either when converted from degrees into grades, and divided by number of grades corresponding to 27 degrees, or when converted from grades into degrees, and divided by number of degrees corresponding to 90 grades.

Let (θ) be the arc.

Then arc in grades $= \theta + \frac{\theta}{9}$ (55) $= \frac{10\theta}{9}$ G.

Also number of grades corresponding to 27 degrees

$$= 27 + \frac{27}{9} = 27 + 3 = 30\,\text{G};$$

$$\therefore\ \frac{10\theta}{9.30} = \frac{\theta}{27} = \text{whole number.}$$

Again, arc in degrees $= \theta - \frac{\theta}{10} = \frac{9\theta}{10} D$ (55),

and number of degrees corresponding to (90) grades

$$= 90 - \frac{90}{10} = 90 - 9 = 81\,D;$$

$$\therefore\ \frac{9\theta}{10.81} = \frac{\theta}{90} = \text{whole number.}$$

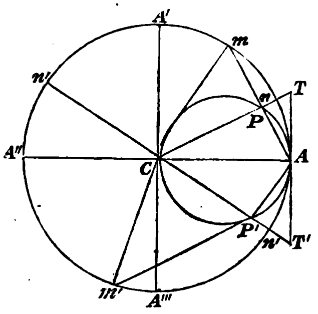
Let $\frac{\theta}{90} = m$. Then $\theta = 90m$;

$\therefore$ by substitution we have $\frac{90m}{27} = \frac{10m}{3} = 3m + \frac{m}{3} =$ whole number.

Let $\frac{m}{3} = n$; $\therefore\ m = 3n$ where (n) may be assumed at pleasure.

$$\begin{array}{lll} \text{Let } n = 1. & \text{Then } m = 3 & \text{and } \theta = 270, \\ n = 2 \ldots\ldots & m = 6 & \text{and } \theta = 540, \\ \&\text{c.} & \&\text{c.} & \&\text{c.} \end{array}$$

174. Find the curve traced out by the intersection of the secant of an arc, and the chord of twice that arc while the arc increases from 0 to a semicircle. Compare its area with that of the given circle.



Let An be the arc of a circle whose radius AC is (r) and $Am = 2An$.

P be the point of intersection of secant CnT and chord Am.

Then when the arc is 0 or n coincides with A, P also coincides with A and the curve is 0.

Now in the two triangles ACP, mCP,

$AC, CP = mC, CP$ and $\angle ACP = \angle mCP$ (Prop. 27. Euclid, III.);

$$\therefore\ \angle mPC = \angle APC = \text{right angle};$$

$\therefore$ the curve which is the locus of P is a semicircle.

(Prop. 31. Euclid, III.).

When n coincides with A', m coincides with A'' and P with C;

$\therefore$ the diameter of this $\frac{1}{2}$ circle is (r).

When n is at any point n' in the *second* quadrant, m will be at some point m' in the *third* quadrant such that $A''m' = 2A'n'$ and point P will be at P' in the fourth quadrant.

When n is at A'' m will be at A and point P at A;

$\therefore$ the curve which is the locus of P while the arc increases from 0 to a semicircle, is the circle whose diameter is the radius of the given circle;

and $\therefore$ area of this circle : area of given circle :: $r^2 : 4r^2$.

(Prop. 2. EUCLID, XII.)

:: 1 : 4;

$$\text{and area of the circle traced out} = \frac{\text{area of given circle}}{4}$$

$$= \frac{(3.141592653590\ldots)\,r^2}{4}$$

$$= (.785398163397\ldots)\,r^2:$$

175. Simplify the expression

$$\frac{3\sin\theta - \sin 3\theta + 3\sin\theta \,.\, \cos 2\theta - \sin 3\theta \,.\, \cos 2\theta}{3\cos\theta + \cos 3\theta - 3\cos\theta \,.\, \cos 2\theta - \cos 3\theta \,.\, \cos 2\theta}$$

$$\frac{3\sin\theta - \sin 3\theta + 3\sin\theta \,.\, \cos 2\theta - \sin 3\theta \,.\, \cos 2\theta}{3\cos\theta + \cos 3\theta - 3\cos\theta \,.\, \cos 2\theta - \cos 3\theta \,.\, \cos 2\theta}$$

$$= \frac{3\sin\theta\,(1 + \cos 2\theta) - \sin 3\theta\,(1 + \cos 2\theta)}{3\cos\theta\,(1 - \cos 2\theta) + \cos 3\theta\,(1 - \cos 3\theta)}$$

$$= \frac{(3\sin\theta - \sin 3\theta) \,.\, (1 + \cos 2\theta)}{(3\cos\theta + \cos 3\theta) \,.\, (1 - \cos 2\theta)}$$

$$= \frac{4\sin^3\theta \,.\, 2\cos^2\theta}{4\cos^3\theta \,.\, 2\sin^2\theta} \quad (32) \text{ and } (34)$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta.$$

176. Simplify the expression

$$(\cot^2\theta + \tan^2\theta) \,.\, (\cot\theta + \tan\theta) - (\cot^2\theta - \tan^2\theta) \,.\, (\cot\theta - \tan\theta).$$

$$(\cot^2\theta + \tan^2\theta) \,.\, (\cot\theta + \tan\theta) - (\cot^2\theta - \tan^2\theta) \,.\, (\cot\theta - \tan\theta)$$

$$= (\cot^2\theta + \tan^2\theta) \,.\, (\cot\theta + \tan\theta) - (\cot\theta + \tan\theta) \,.\, (\cot\theta - \tan\theta)^2$$

$$= \{\cot^2\theta + \tan^2\theta - (\cot\theta - \tan\theta)^2\} \,.\, (\cot\theta + \tan\theta)$$

$$= \{\cot^2\theta + \tan^2\theta - (\cot^2\theta - 2\tan\theta \,.\, \cot\theta + \tan^2\theta)\} \,.\, (\cot\theta + \tan\theta)$$

$$= 2\tan\theta \,.\, \cot\theta \,.\, (\cot\theta + \tan\theta)$$

$$= 2 \,.\, (\cot\theta + \tan\theta)$$

$$= 2\,\text{cosec}\, 2\theta.$$

177. If $\frac{\cot^2\theta+1}{\cot^2\theta-1}=2\operatorname{cosec}^2\phi-1$, then $\tan\theta=\cos\phi$.

$$\frac{\cot^2\theta+1}{\cot^2\theta-1}=2\operatorname{cosec}^2\phi-1$$

$$=\operatorname{cosec}^2\phi+\cot^2\phi$$

$$=\frac{\sec^2\phi}{\tan^2\phi}+\frac{1}{\tan^2\phi}\quad(23)\text{ and }(25)$$

$$=\frac{\sec^2\phi+1}{\sec^2\phi-1};$$

$$\therefore\ \cot^2\theta+1\ :\ \cot^2\theta-1\ ::\ \sec^2\phi+1\ :\ \sec^2\phi-1,$$

$$\text{and }\therefore\ 2\cot^2\theta\ :\ 2\ ::\ 2\sec^2\phi\ :\ 2;$$

$$\therefore\ 2\cot^2\theta=2\sec^2\phi,$$

$$\text{and }\cot\theta=\sec\phi;$$

$$\therefore\ \tan\theta=\frac{1}{\cot\theta}=\frac{1}{\sec\phi}=\cos\phi.$$

178. If $\frac{\tan^2\theta-(1+2\tan^3\theta)}{\tan^3\theta-\tan\theta}=(m+1)\,.\,(m-1)$, then

$$\sec 2\theta+\operatorname{cosec}2\theta+\cot 2\theta=m^2.$$

Here $\tan^2\theta-1-2\tan^3\theta=m^2\,.\,(\tan^3\theta-\tan\theta)-\tan^3\theta+\tan\theta$;

$$\therefore\ m^2\,.\,(\tan\theta-\tan^3\theta)=\tan\theta+\tan^3\theta+1-\tan^2\theta;$$

$$\therefore\ m^2=\frac{\tan\theta+\tan^3\theta}{\tan\theta-\tan^3\theta}+\frac{1-\tan^2\theta}{\tan\theta-\tan^3\theta}$$

$$=\frac{1+\tan^2\theta}{1-\tan^2\theta}+\frac{1}{\tan\theta}$$

$$=\frac{1}{\cos 2\theta}+\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\quad(30)\text{ and }(31)$$

$$=\frac{1}{\cos 2\theta}+\frac{1+\cos 2\theta}{\sin 2\theta}$$

$$=\frac{1}{\cos 2\theta}+\frac{1}{\sin 2\theta}+\frac{\cos 2\theta}{\sin 2\theta}$$

$$=\sec 2\theta+\operatorname{cosec}2\theta+\cot 2\theta.$$

179. If $\dfrac{\tan(30+\theta)}{\tan(60+\theta)} = \tan 2\phi - \sec 2\phi$, then $\tan\phi = 2\cos 2\theta$.

$$\frac{\tan(30+\theta)}{\tan(60+\theta)} = \frac{\dfrac{1}{\sqrt{3}} + \tan\theta}{1 - \dfrac{1}{\sqrt{3}}.\tan\theta} \times \frac{1-\sqrt{3}.\tan\theta}{\sqrt{3}+\tan\theta} \quad (19), (20) \text{ and } (38)$$

$$= \frac{1+\sqrt{3}.\tan\theta}{\sqrt{3}-\tan\theta} \times \frac{1-\sqrt{3}.\tan\theta}{\sqrt{3}+\tan\theta}$$

$$= \frac{1-3\tan^2\theta}{3-\tan^2\theta}$$

$$= \frac{1-\left(\dfrac{3-3\cos 2\theta}{1+\cos 2\theta}\right)}{3-\left(\dfrac{1-\cos 2\theta}{1+\cos 2\theta}\right)} \quad (31)$$

$$= \frac{2\cos 2\theta - 1}{2\cos 2\theta + 1}.$$

Again, $\tan 2\phi - \sec 2\phi = \cot\left(\frac{\pi}{2} - 2\phi\right) - \operatorname{cosec}\left(\frac{\pi}{2} - 2\phi\right)$

$$= -\tan\left(\frac{\pi}{2} - \phi\right) \quad (28)$$

$$= \frac{\tan\phi - 1}{\tan\phi + 1} \quad (12) \text{ and } (38);$$

$$\therefore \frac{2\cos 2\theta - 1}{2\cos 2\theta + 1} = \frac{\tan\phi - 1}{\tan\phi + 1};$$

$$\therefore 2\cos 2\theta + 1 : 2\cos 2\theta - 1 :: \tan\phi + 1 : \tan\phi - 1;$$

$$\therefore 4\cos 2\theta : 2 :: 2\tan\phi : 2;$$

$$\therefore 4\cos 2\theta = 2\tan\phi,$$

$$\text{and } 2\cos 2\theta = \tan\phi.$$

180. Exhibit $\dfrac{\tan\frac{3\pi}{8} + \tan\frac{\pi}{8}}{\tan\frac{3\pi}{8} - \tan\frac{\pi}{8}}$ as a continued fraction.

$$\frac{\tan\frac{3\pi}{8}+\tan\frac{\pi}{8}}{\tan\frac{3\pi}{8}-\tan\frac{\pi}{8}}=\frac{\cot\frac{\pi}{8}+\tan\frac{\pi}{8}}{\cot\frac{\pi}{8}-\tan\frac{\pi}{8}} \quad (5)$$

$$=\frac{2\operatorname{cosec}\frac{\pi}{4}}{2\cot\frac{\pi}{4}} \quad (27)$$

$$=\sec\frac{\pi}{4} \quad (23) \text{ and } (25)$$

$$=\sqrt{2} \quad (16)$$

$$=1+\sqrt{2}-1$$

$$=1+\frac{1}{\sqrt{2}+1}$$

$$=1+\frac{1}{2+\sqrt{2}-1}$$

$$=1+\cfrac{1}{2+\cfrac{1}{\sqrt{2}+1}}$$

$$=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots\dots\dots}}}}$$

181. Compare the vanishing fractions

$$u_1=\frac{\sec\theta-\operatorname{cosec}\theta}{\sin\theta-\cos\theta}, \qquad u_2=\frac{\cot\theta-\tan\theta}{\operatorname{cosec}\theta-\sec\theta}, \qquad \text{when } \theta=\frac{\pi}{4}.$$

$$u_1=\frac{\sec\theta-\operatorname{cosec}\theta}{\sin\theta-\cos\theta}$$

$$=\frac{\sec\theta\,.\sin\theta\,.\cos\theta-\operatorname{cosec}\theta\,.\sin\theta\,.\cos\theta}{(\sin\theta-\cos\theta)\,.\sin\theta\,.\cos\theta}$$

$$=\frac{\sin\theta-\cos\theta}{\sin\theta\,.\cos\theta\,.(\sin\theta-\cos\theta)}=\frac{1}{\sin\theta\,.\cos\theta}$$

$$=\sec\frac{\pi}{4}\,.\operatorname{cosec}\frac{\pi}{4}$$

$$=2 \quad (16).$$

$$\text{Similarly } u_2 = \cos\theta + \sin\theta$$
$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$
$$= \frac{2}{\sqrt{2}} \quad (15)$$
$$= \sqrt{2};$$
$$\therefore\ u_1 : u_2 :: 2 : \sqrt{2}$$
$$:: \sqrt{2} : 1.$$

182. If $\theta + \phi + \psi + \chi = \pi$, then

$$\frac{\sin . \{2 . (\theta - \phi)\} . \sin(\psi + \chi)}{\cos(\theta - \phi)} + \frac{\sin\{2 . (\phi - \psi)\} . \sin\{(\theta + \chi)\}}{\cos(\phi - \psi)}$$
$$+ \frac{\sin\{2 . (\psi - \chi)\} . \sin(\theta + \phi)}{\cos(\psi - \chi)} + \frac{\sin\{2 . (\chi - \theta)\} . \sin(\phi + \psi)}{\cos(\chi - \theta)} = 0.$$

$$\frac{\sin\{2 . (\theta - \phi)\}}{\cos(\theta - \phi)} = \frac{2\sin(\theta - \phi) . \cos(\theta - \phi)}{\cos(\theta - \phi)} = 2\sin(\theta - \phi),$$

and $\sin(\psi + \chi) = \sin\{\pi - (\theta + \phi)\} = \sin(\theta + \phi)$;

$$\therefore\ \frac{\sin\{2 . (\theta - \phi)\} \sin(\psi + \chi)}{\cos(\theta - \phi)} = 2\sin(\theta + \phi) . \sin(\theta - \phi) = \sin^2\theta - \sin^2\phi \quad (41).$$

$$\text{Similarly } \frac{\sin\{2 . (\phi - \psi)\} . \sin(\theta + \chi)}{\cos(\phi - \psi)} = \sin^2\phi - \sin^2\psi,$$
$$\frac{\sin\{2 . (\psi - \chi)\} . \sin(\theta + \phi)}{\cos(\psi - \chi)} = \sin^2\psi - \sin^2\chi,$$
$$\frac{\sin\{2 . (\chi - \theta)\} . \sin(\phi + \psi)}{\cos(\chi - \theta)} = \sin^2\chi - \sin^2\theta;$$

$$\therefore\ \frac{\sin\{2 . (\theta - \phi)\} . \sin(\psi - \chi)}{\cos(\theta - \phi)} + \frac{\sin\{2 . (\phi - \psi)\} . \sin(\theta + \chi)}{\cos(\phi - \psi)}$$
$$+ \frac{\sin\{2 . (\psi - \chi)\} . \sin(\theta + \phi)}{\cos(\psi - \chi)} + \frac{\sin\{2 . (\chi - \theta)\} . \sin(\phi + \psi)}{\cos(\chi - \theta)} = 0.$$

Cor. Precisely in the same way it may be shewn that the proposition holds good if $\theta + \phi + \psi + \chi = 2\pi$, or if θ, ϕ, ψ and χ be the angles of any quadrilateral figure.

183. Extract the square root of $\cos 2\theta \pm \sqrt{-1}\,.\,\sin 2\theta$ independently of Demoivre's formula.

Let $\sqrt{(\cos 2\theta \pm \sqrt{-1}\,.\,\sin 2\theta)} = \sqrt{x} \pm \sqrt{y}$;

$$\therefore\ \cos 2\theta \pm \sqrt{-1}\,.\,\sin 2\theta = x + y \pm 2\sqrt{xy};$$

$\therefore\ x + y = \cos 2\theta$ and $\sqrt{-1}\,.\,\sin 2\theta = 2\sqrt{xy}$. Hind's *Algebra*, 120;

$$\therefore\ x^2 + 2xy + y^2 = \cos^2 2\theta,$$

and $4xy = -\sin^2 2\theta$;

$$\therefore\ x^2 - 2xy + y^2 = \cos^2 2\theta + \sin^2 2\theta = 1;$$

$$\therefore\ x - y = 1,$$

and $x + y = \cos 2\theta$;

$$\therefore\ 2x = 1 + \cos 2\theta = 2\cos^2\theta; \quad \therefore\ \sqrt{x} = \cos\theta,$$

$$2y = -(1 - \cos 2\theta) = -2\sin^2\theta; \quad \therefore\ \sqrt{y} = \sqrt{-1}\,.\,\sin\theta;$$

$$\therefore\ \sqrt{(\cos 2\theta \pm \sqrt{-1}\,.\,\sin 2\theta)} = \cos\theta \pm \sqrt{-1}\,.\,\sin\theta.$$

184. Extract the cube root of $\cos 3\theta \pm \sqrt{-1}\,.\,\sin 3\theta$.

Let $\sqrt[3]{(\cos 3\theta + \sqrt{-1}\,.\,\sin 3\theta)} = x + \sqrt{y}$.

Then $\sqrt[3]{(\cos 3\theta - \sqrt{-1}\,.\,\sin 3\theta)} = x - \sqrt{y}$. Hind's *Algebra*, 124.

$$\therefore\ \sqrt[3]{(\cos^2 3\theta + \sin^2 3\theta)} = x^2 - y,$$

or $1 = x^2 - y$ and $y = x^2 - 1$.

Again $\cos 3\theta + \sqrt{-1}\,.\,\sin 3\theta = x^3 + 3x^2\,.\,\sqrt{y} + 3xy + y\sqrt{y}$

$$= (x^3 + 3xy) + (3x^2 + y)\sqrt{y};$$

$\therefore\ \cos 3\theta = x^3 + 3xy$. Hind's *Algebra*, 120.

$$= x^3 + 3x\,.\,(x^2 - 1)$$

$$= x^3 + 3x^3 - 3x,$$

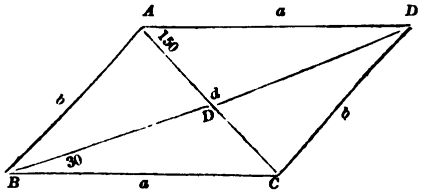
or $4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$ (34);

$$\therefore\ x = \cos\theta; \quad \therefore\ y = \cos^2\theta - 1 = -\sin^2\theta;$$

$$\therefore\ \sqrt{y} = \sqrt{-1}\,.\,\sin\theta;$$

$$\therefore\ \sqrt[3]{(\cos 3\theta \pm \sqrt{-1}\,.\,\sin 3\theta)} = \cos\theta \pm \sqrt{-1}\,.\,\sin\theta.$$

185. Required the nature of that parallelogram one of whose angles is 150°, and its diagonals such that the square of their difference is equal to twice their product.



$ABCD$ is the parallelogram where $\angle BAD = 150$.

Let AD, AB be denoted by a, b and BD, AC by D, d.

Then BC will also be denoted by a, and

$$\angle ABC = 180 - 150 = 30.$$

$$\text{Now } 2Dd = (D-d)^2 = D^2 - 2Dd + d^2;$$

$$\therefore\ D^2 + d^2 = 4Dd.$$

$$\text{But } a^2 + b^2 - 2ab \,.\, \cos 150 = D^2 \quad (50),$$

$$\text{or } a^2 + b^2 + 2ab \,.\, \cos 30 = D^2 \quad (4),$$

$$\text{and } a^2 + b^2 - 2ab \,.\, \cos 30 = d^2 \quad (50);$$

$$\therefore\ 2a^2 + 2b^2 = D^2 + d^2 = 4Dd;$$

$$\therefore\ a^2 + b^2 = 2Dd;$$

$$\therefore\ \frac{a^2 + b^2}{2} = Dd;$$

$$\therefore\ \frac{a^4 + 2a^2b^2 + b^4}{4} = D^2d^2 = (a^2 + b^2)^2 - 4a^2b^2 \,.\, \cos^2 30$$

$$= a^4 + 2a^2b^2 + b^4 - 3a^2b^2 \quad (17)$$

$$= a^4 - a^2b^2 + b^4;$$

$$\therefore\ a^4 + 2a^2b^2 + b^4 = 4a^4 - 4a^2b^2 + 4b^4;$$

$$\therefore\ 3 \,.\, (a^4 + b^4) - 6a^2b^2 = 0;$$

$$\therefore\ a^4 - 2a^2b^2 + b^4 = 0;$$

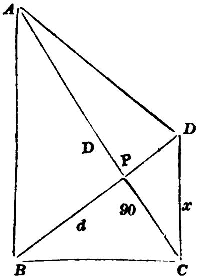
$$\therefore\ a^2 - b^2 = 0;$$

$$\therefore\ a^2 = b^2,$$

$$\text{and } a = b;$$

therefore the parallelogram is a rhombus.

186. *ABCD* is a trapezium whose angles *ABC, BCD* are right angles, and whose diagonals *AC, BD* also intersect each other at right angles in *P*. Find the sides and the area of *ABCD* and the sine and cosine of $\angle ACD$.



Let $CD = x$, $AC = D$, $BD = d$. Then $BC = \sqrt{(d^2 - x^2)}$.

Now $\dfrac{x}{d} = \dfrac{\sin CBD}{\sin BCD} = \sin CBD = \cos BDC = \sin ACD = \cos ACB$

$$= \frac{BC}{AC} = \frac{\sqrt{(d^2 - x^2)}}{D};$$

$$\therefore \frac{x^2}{d^2} = \frac{d^2 - x^2}{D^2}; \quad \therefore D^2 x^2 = d^4 - d^2 x^2;$$

$$\therefore (D^2 + d^2)\, x^2 = d^4; \quad \therefore x^2 = \frac{d^4}{D^2 + d^2} \text{ and } x = \frac{d^2}{\sqrt{D^2 + d^2}} = CD \quad (1);$$

$$\therefore BC = \sqrt{d^2 - x^2} = \frac{Dd}{\sqrt{D^2 + d^2}} \quad (2);$$

$$\therefore AB = \sqrt{\left(D^2 - \frac{D^2 d^2}{D^2 + d^2}\right)} = \frac{D^2}{\sqrt{(D^2 + d^2)}} \quad (3).$$

$$\text{Also } \cos ACD = \sin ACB = \frac{AB}{AC} = \frac{D}{\sqrt{(D^2 + d^2)}} \quad (4),$$

$$\text{and } \sin ACD = \frac{\sqrt{(d^2 - x^2)}}{D} = \frac{\sqrt{\left(d^2 - \dfrac{d^4}{D^2 + d^2}\right)}}{D} = \frac{d}{\sqrt{(D^2 + d^2)}} \quad (5);$$

$$\therefore AD^2 = AC^2 + CD^2 - 2\,AC \,.\, CD \,.\, \cos ACD \quad (50)$$

$$= D^2 + \frac{d^4}{D^2 + d^2} - 2\,D \,.\, \frac{d^2}{\sqrt{(D^2 + d^2)}} \,.\, \frac{D}{\sqrt{(D^2 + d^2)}}$$

$$= \frac{D^4 + D^2d^2 + d^4 - 2D^2d^2}{D^2 + d^2};$$

$$\therefore\ AD = \sqrt{\left\{\frac{(D^2 - d^2)^2 + D^2d^2}{D^2 + d^2}\right\}} \quad (6).$$

And area = triangle ABC + triangle ACD

$$= \frac{AB \,.\, AC}{2} \,.\, \sin BAC + \frac{AC \,.\, CD}{2} \,.\, \sin ACD \quad (49)$$

$$= \frac{AC \,.\, \sin ACD}{2} \,.\, (AB + CD)$$

$$= \frac{D}{2} \,.\, \frac{d}{\sqrt{(D^2 + d^2)}} \left\{\frac{D^2}{\sqrt{(D^2 + d^2)}} + \frac{d^2}{\sqrt{(D^2 + d^2)}}\right\}$$

$$= \frac{Dd}{2\sqrt{(D^2 + d^2)}} \left\{\frac{D^2 + d^2}{\sqrt{(D^2 + d^2)}}\right\}$$

$$= \frac{Dd}{2} \,.\, \left(\frac{D^2 + d^2}{D^2 + d^2}\right)$$

$$= \frac{Dd}{2} \quad (7).$$

187. Given the logarithms of 2 and 3. Find the logarithm of 5184.

$$5184 = 8 \times 648 = 8 \times 8 \times 81 = 8 \times 8 \times 9 \times 9 = 8^2 \times 9^2$$

$$= (2^3)^2 \times (3^2)^2$$

$$= 2^6 \times 3^4;$$

$$\therefore\ \log 5184 = \log (2^6 \times 3^4)$$

$$= \log 2^6 + \log 3^4 \quad (72)$$

$$= 6 \log 2 + 4 \log 3 \quad (71).$$

188. If $\theta = \sin^{-1}\dfrac{\sqrt{2 + \sqrt{2}}}{2}$ and $\phi = \sin^{-1}\dfrac{\sqrt{2 - \sqrt{2}}}{2}$. Then $\theta + \phi = \dfrac{\pi}{2}$.

Here $\sin\theta = \dfrac{\sqrt{2 + \sqrt{2}}}{2}$ and $\sin\phi = \dfrac{\sqrt{2 - \sqrt{2}}}{2}$;

$\therefore\ \sin^2\theta = \dfrac{2 + \sqrt{2}}{4}$; and $\therefore\ \cos^2\theta = 1 - \dfrac{2 + \sqrt{2}}{4} = \dfrac{2 - \sqrt{2}}{4}$;

$$\therefore\ \cos\theta = \frac{\sqrt{2-\sqrt{2}}}{2} = \sin\phi = \cos\left(\frac{\pi}{2} - \phi\right);$$

$$\therefore\ \theta = \frac{\pi}{2} - \phi \quad \text{and} \quad \theta + \phi = \frac{\pi}{2}.$$

189. If $\theta = \cos^{-1}\frac{3}{4}$, $\phi = \cos^{-1}\frac{9}{16}$ and $\psi = \cos^{-1}\frac{1}{8}$;

then $\theta + \phi + \psi = \pi$.

Here $\cos\theta = \frac{3}{4}$; and $\therefore\ \sin^2\theta = 1 - \frac{9}{16} = \frac{7}{16}$ and $\sin\theta = \frac{\sqrt{7}}{4}$,

$\cos\phi = \frac{9}{16}$; $\therefore\ \sin^2\phi = 1 - \frac{81}{256} = \frac{175}{256}$ and $\sin\phi = \frac{5\sqrt{7}}{16}$;

$$\therefore\ \cos(\theta+\phi) = \cos\theta\,.\,\cos\phi - \sin\theta\,.\,\sin\phi = \frac{3}{4}\,.\,\frac{9}{16} - \frac{\sqrt{7}}{4}\,.\,\frac{5\sqrt{7}}{16} = \frac{-1}{8};$$

$$\text{and } \therefore\ \cos\psi = \frac{1}{8} = -\cos(\theta+\phi) = \cos\{\pi - (\theta+\phi)\} \quad (4);$$

$$\therefore\ \psi = \pi - (\theta+\phi);$$

$$\therefore\ \theta + \phi + \psi = \pi.$$

190. $\sec^{-1}\sqrt{2} - \sec^{-1}\frac{2\sqrt{2}}{\sqrt{3}+1} = 30°$.

$$\sec^{-1}\sqrt{2} - \sec^{-1}\frac{2\sqrt{2}}{\sqrt{3}+1} = \sec^{-1}\left\{\frac{\sqrt{2}\,.\,\frac{2\sqrt{2}}{\sqrt{3}+1}}{1+\sqrt{\{(\sqrt{2})^2-1\}\,.\left\{\left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right)^2-1\right\}}}\right\} \quad (39)$$

$$= \sec^{-1}\left(\frac{\frac{4}{\sqrt{3}+1}}{1+2-\sqrt{3}}\right)$$

$$= \sec^{-1}\left\{\frac{4}{(\sqrt{3}+1)\,.\,(3-\sqrt{3})}\right\}$$

$$= \sec^{-1}\left\{\frac{4}{(\sqrt{3}+1)\,.\,\sqrt{3}\,.\,(\sqrt{3}-1)}\right\}$$

$$= \sec^{-1}\left(\frac{4}{2\sqrt{3}}\right)$$

$$= \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$= 30 \quad (18).$$

191. $\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3} - \cot^{-1}\frac{1}{2+\sqrt{3}} = 60.$

$$\cot^{-1}\frac{1}{2+\sqrt{3}} = \cot^{-1}\left\{\frac{2-\sqrt{3}}{(2+\sqrt{3}).(2-\sqrt{3})}\right\} = \cot^{-1}(2-\sqrt{3}),$$

$$\text{and } \cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3} = \cot^{-1}\left(\frac{\frac{1}{2}\cdot\frac{1}{3}-1}{\frac{1}{3}+\frac{1}{2}}\right) = \cot^{-1}(-1) \quad (38)$$

$$\therefore\ \cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3} - \cot^{-1}\frac{1}{2+\sqrt{3}} = \cot^{-1}(-1) - \cot^{-1}(2-\sqrt{3})$$

$$= \cot^{-1}\left\{\frac{-1.(2-\sqrt{3})+1}{(2-\sqrt{3})+1}\right\} \quad (38)$$

$$= \cot^{-1}\left(\frac{\sqrt{3}-1}{3-\sqrt{3}}\right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 60 \quad (20)$$

192. $\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{26} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{65} = 45,$

$$\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{26} = \operatorname{cosec}^{-1}\left\{\frac{\sqrt{10}.\sqrt{26}}{\sqrt{(\sqrt{26})^2-1}+\sqrt{(\sqrt{10})^2-1}}\right\} \quad (39)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{10}.\sqrt{26}}{5+3}\right)$$

$$= \operatorname{cosec}^{-1}\frac{\sqrt{65}}{4}.$$

Again,

$$\operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{65} = \operatorname{cosec}^{-1}\left\{\frac{\sqrt{65}.\sqrt{50}}{\sqrt{(\sqrt{65})^2-1}+\sqrt{(\sqrt{50})^2-1}}\right\} \quad (39)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{130}}{3}\right);$$

$$\therefore\ \operatorname{cosec}^{-1}(\sqrt{10}) + \operatorname{cosec}^{-1}(\sqrt{26}) + \operatorname{cosec}^{-1}(\sqrt{50}) + \operatorname{cosec}^{-1}(\sqrt{65})$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{65}}{4}\right) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{130}}{3}\right)$$

$$= \operatorname{cosec}^{-1}\left\{\frac{\frac{\sqrt{130}}{3}\cdot\frac{\sqrt{65}}{4}}{\sqrt{\left(\frac{\sqrt{130}}{3}\right)^2 - 1} + \sqrt{\left(\frac{\sqrt{65}}{4}\right)^2 - 1}}\right\} \quad (39)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\frac{\sqrt{65}}{4}\cdot\frac{\sqrt{130}}{3}}{\frac{11}{3} + \frac{7}{4}}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{65\sqrt{2}}{65}\right)$$

$$= \operatorname{cosec}^{-1}(\sqrt{2})$$

$$= 45 \quad (16).$$

193. $\tan^{-1}(x + 1) = 3\tan^{-1}(x - 1)$. Determine (x).

$$\tan^{-1}(x + 1) = 3\tan^{-1}(x - 1)$$

$$= \tan^{-1}\left\{\frac{(x-1)^3 - 3(x-1)}{3(x-1)^2 - 1}\right\} \quad (35);$$

$$\therefore\ (x + 1) = \frac{(x-1)^3 - 3(x-1)}{3(x-1)^2 - 1}$$

$$= \frac{(x-1)^2 - 3}{3(x-1) - \frac{1}{(x-1)}};$$

$$\therefore\ 3(x^2 - 1) - \frac{x+1}{x-1} = (x-1)^2 - 3;$$

$$\therefore\ 3x^2 - \frac{x+1}{x-1} = (x-1)^2;$$

$$\therefore\ 3x^3 - 3x^2 - x - 1 = x^3 - 3x^2 + 3x - 1;$$

$$\therefore\ 2x^3 = 4x;$$

$$\therefore\ x^2 = 2,$$

$$\text{and } x = \pm\sqrt{2}.$$

194. $\text{chd}^2\theta . \text{cosec}\,\theta = 2\tan\frac{\theta}{2}$. Prove this by the expressions for sine and cosine of an arc, in terms of the exponential functions of that arc;

$$\text{chd}\,\theta = 2\sin\frac{\theta}{2} = 2\left(\frac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{2\sqrt{-1}}\right) \quad (58);$$

$$\therefore\ \text{chd}^2\theta = \frac{1}{(\sqrt{-1})^2} . \left(\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}\right)^2,$$

$$\text{and } \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{2\sqrt{-1}}{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}$$

$$= \frac{2\sqrt{-1}}{\left(\epsilon^{\frac{\theta}{2}\sqrt{-1}} + \epsilon^{-\frac{\theta}{2}\sqrt{-1}}\right) . \left(\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}\right)};$$

$$\therefore\ \text{chd}^2\theta . \text{cosec}\,\theta = \frac{2}{\sqrt{-1}} . \left(\frac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{\epsilon^{\frac{\theta}{2}\sqrt{-1}} + \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}\right)$$

$$= 2\left\{\frac{\dfrac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{2\sqrt{-1}}}{\dfrac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} + \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{2}}\right\}$$

$$= 2\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) \quad (58)$$

$$= 2\tan\frac{\theta}{2}.$$

195. $2\,\text{cosec}\,2\theta = \frac{\sec\theta + \text{cosec}\,\theta}{\sin\theta + \cos\theta}$. Prove this by the impossible exponential functions of θ.

$$2\,\text{cosec}\,2\theta = \cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\dfrac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2}}{\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}} + \frac{\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}}{\dfrac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2}} \quad (58)$$

$$= \frac{\dfrac{\epsilon^{2\theta\sqrt{-1}} + 2 + \epsilon^{-2\theta\sqrt{-1}}}{4} - \left(\dfrac{\epsilon^{2\theta\sqrt{-1}} - 2 + \epsilon^{-2\theta\sqrt{-1}}}{4}\right)}{\left(\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}\right) \cdot \left(\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2}\right)}$$

$$= \frac{1}{\left(\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}\right) \cdot \left(\dfrac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2}\right)}$$

$$= \frac{\dfrac{2\sqrt{-1}}{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}} + \dfrac{2}{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}}{\dfrac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2} + \dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}}$$

$$= \frac{\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}}{\cos\theta + \sin\theta} \quad (58)$$

$$= \frac{\operatorname{cosec}\theta + \sec\theta}{\cos\theta + \sin\theta}.$$

196. $$\frac{\tan\theta}{\operatorname{chd}\theta} = \epsilon^{\frac{\theta}{2}\sqrt{-1}} \left(\frac{\epsilon^{\theta\sqrt{-1}} + 1}{\epsilon^{2\theta\sqrt{-1}} + 1}\right).$$ Required proof.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}}{\dfrac{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}}{2}} = \frac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{\sqrt{-1}\,(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}})}$$

$$= \frac{\epsilon^{2\theta\sqrt{-1}} - 1}{\sqrt{-1} \,.\, (\epsilon^{2\theta\sqrt{-1}} + 1)} = \frac{(\epsilon^{\theta\sqrt{-1}} + 1) \,.\, (\epsilon^{\theta\sqrt{-1}} - 1)}{\sqrt{-1} \,.\, (\epsilon^{2\theta\sqrt{-1}} + 1)},$$

$$\operatorname{chd}\theta = 2\sin\frac{\theta}{2} = \frac{\epsilon^{\frac{\theta}{2}\sqrt{-1}} - \epsilon^{-\frac{\theta}{2}\sqrt{-1}}}{\sqrt{-1}} = \frac{\epsilon^{\theta\sqrt{-1}} - 1}{\sqrt{-1} \,.\, \epsilon^{\frac{\theta}{2}\sqrt{-1}}};$$

$$\therefore \frac{\tan\theta}{\operatorname{chd}\theta} = \epsilon^{\frac{\theta}{2}\sqrt{-1}} \left(\frac{\epsilon^{\ \sqrt{-1}} + 1}{\epsilon^{2\theta\sqrt{-1}} + 1}\right).$$

197. Apply Demoivre's formula to shew that

$$\sqrt[n]{(\cos 2n\theta \pm \sqrt{-1} \,.\, \sin 2n\theta)} = \sqrt[m]{(\cos 2m\theta \pm \sqrt{-1} \,.\, \sin 2m\theta)}$$

$$\cos 2n\theta \pm \sqrt{-1} \,.\, \sin 2n\theta = (\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^{2n} \quad (56)$$

$$= \{(\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^2\}^n;$$

$$\therefore \sqrt{(\cos 2n\theta \pm \sqrt{-1} \,.\, \sin 2n\theta)} = (\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^2.$$

Similarly $\cos 2m\theta \pm \sqrt{-1} \,.\, \sin 2m\theta = \{(\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^2\}^m$;

$$\therefore \sqrt[m]{(\cos 2m\theta \pm \sqrt{-1} \,.\, \sin 2m\theta)} = (\cos\theta \pm \sqrt{-1} \,.\, \sin\theta)^2$$

$$= \sqrt{(\cos 2n\theta \pm \sqrt{-1} \,.\, \sin 2n\theta)}.$$

COR. Let $m = 3$ and $n = 4$, and we shall have

$$\sqrt[3]{(\cos 6\theta \pm \sqrt{-1} \,.\, \sin 6\theta)} = \sqrt[4]{(\cos 8\theta \pm \sqrt{-1} \,.\, \sin 8\theta)}.$$

198. If $\dfrac{\cos(\theta - \psi)}{\cos(\theta + \psi)} = \sqrt{\left(\dfrac{\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi}{\cos 2\phi - \sqrt{-1} \,.\, \sin 2\phi}\right)}$.

$$\text{Then} \quad \tan\theta \,.\, \tan\psi = \frac{\epsilon^{2\phi\sqrt{-1}} - 1}{\epsilon^{2\phi\sqrt{-1}} + 1},$$

$$\frac{\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi}{\cos 2\phi - \sqrt{-1} \,.\, \sin 2\phi}$$

$$= \frac{(\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi)^2}{(\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi) \,.\, (\cos 2\phi - \sqrt{-1} \,.\, \sin 2\phi)}$$

$$= \frac{(\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi)^2}{\cos^2 2\phi + \sin^2 2\phi};$$

$$\therefore \sqrt{\left(\frac{\cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi}{\cos 2\phi - \sqrt{-1} \,.\, \sin 2\phi}\right)} = \cos 2\phi + \sqrt{-1} \,.\, \sin 2\phi$$

$$= (\cos\phi + \sqrt{-1} \,.\, \sin\phi)^2 \quad (56);$$

$$\therefore \cos\phi + \sqrt{-1} \,.\, \sin\phi = \sqrt{\left\{\frac{\cos(\theta - \psi)}{\cos(\theta + \psi)}\right\}};$$

$$\text{and } \therefore \cos\phi - \sqrt{-1} \,.\, \sin\phi = \sqrt{\left\{\frac{\cos(\theta + \psi)}{\cos(\theta - \psi)}\right\}};$$

$$\therefore 2\cos\phi = \sqrt{\left\{\frac{\cos(\theta - \psi)}{\cos(\theta + \psi)}\right\}} + \sqrt{\left\{\frac{\cos(\theta + \psi)}{\cos(\theta - \psi)}\right\}},$$

and $2\sqrt{-1}\,.\,\sin\phi = \sqrt{\left\{\dfrac{\cos(\theta-\psi)}{\cos(\theta+\psi)}\right\}} - \sqrt{\left\{\dfrac{\cos(\theta+\psi)}{\cos(\theta-\psi)}\right\}}$.

$$\text{But } \frac{\epsilon^{2\phi\sqrt{-1}}-1}{\epsilon^{2\phi\sqrt{-1}}+1} = \frac{\epsilon^{\phi\sqrt{-1}}-\epsilon^{-\phi\sqrt{-1}}}{\epsilon^{\phi\sqrt{-1}}+\epsilon^{-\phi\sqrt{-1}}} = \frac{2\sqrt{-1}\,.\,\sin\phi}{2\cos\phi} \quad (58)$$

$$= \frac{\sqrt{\left\{\dfrac{\cos(\theta-\psi)}{\cos(\theta+\psi)}\right\}} - \sqrt{\left\{\dfrac{\cos(\theta+\psi)}{\cos(\theta-\psi)}\right\}}}{\sqrt{\left\{\dfrac{\cos(\theta-\psi)}{\cos(\theta+\psi)}\right\}} + \sqrt{\left\{\dfrac{\cos(\theta+\psi)}{\cos(\theta-\psi)}\right\}}}$$

$$= \frac{\cos(\theta-\psi)-\cos(\theta+\psi)}{\cos(\theta-\psi)+\cos(\theta+\psi)}$$

$$= \frac{2\sin\theta\,.\,\sin\psi}{2\cos\theta\,.\,\cos\psi} \quad (42)$$

$$= \tan\theta\,.\,\tan\psi.$$

199. Prove that

$$\cos\frac{\theta}{2}\,.\,\cos\frac{\theta}{4}\,.\,\cos\frac{\theta}{8}\ldots\ldots = 1, \text{ when } \theta = 0.$$

$$\cos\frac{\theta}{2}\,.\,\cos\frac{\theta}{4}\,.\,\cos\frac{\theta}{8}\ldots\ldots = \frac{1}{\sec\frac{\theta}{2}\,.\,\sec\frac{\theta}{4}\,.\,\sec\frac{\theta}{8}\ldots\ldots}$$

$$= \frac{\sin\theta}{\sin\theta\,.\,\sec\frac{\theta}{2}\,.\,\sec\frac{\theta}{4}\,.\,\sec\frac{\theta}{8}\ldots\ldots}$$

$$= \frac{\sin\theta}{\theta} \quad (60)$$

$$= 1, \text{ when } \theta = 0.$$

200. $\tan\theta = \dfrac{\theta^2 + \dfrac{\theta^4}{1.3} + \dfrac{2\theta^6}{1.3.5} + \ldots\ldots}{\dfrac{3^{\frac{5}{2}}}{5} - \dfrac{3^{\frac{7}{2}}}{7} + \dfrac{3^{\frac{9}{2}}}{9} - \ldots\ldots}$ What is the value of (θ)?

Here $\cot\theta\,.\left(\theta^2 + \dfrac{\theta^4}{1.3} + \dfrac{2\theta^6}{1.3.5} + \ldots\ldots\right) = \dfrac{1}{5}3^{\frac{5}{2}} - \dfrac{1}{7}3^{\frac{7}{2}} + \dfrac{1}{9}3^{\frac{9}{2}} - \ldots\ldots$

$$\text{or } \cot\theta \,.\, \tan\theta \,.\, \theta = 3^{\frac{1}{2}} - \frac{1}{3}3^{\frac{3}{2}} + \frac{1}{5}3^{\frac{5}{2}} - \frac{1}{7}3^{\frac{7}{2}} + \ldots (64);$$

$$\therefore\ \theta = \tan 60 - \frac{1}{3}\tan^3 60 + \frac{1}{5}\tan^5 60 - \frac{1}{7}\tan^7 60 + \ldots (19)$$

$$= 60^{\circ} \quad (59).$$

201. Prove that

$$\cos\theta \,.\, \cos 2\theta \,.\, \cos 4\theta \,.\, \cos 8\theta \ldots\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n \,.\, \theta}{2^n \,.\, \sin\theta},$$

and apply it to find θ in the equation,

$$\sin 8\theta \,.\, \cos 8\theta \,.\, \cos 16\theta \ldots\ldots \cos 256\theta = \left(\frac{\sqrt{-1}}{8}\right)^2,$$

$$\cos\theta = \frac{2\sin\theta \,.\, \cos\theta}{2\sin\theta} = \frac{\sin 2\theta}{2\sin\theta},$$

$$\cos 2\theta = \frac{\sin 4\theta}{2\sin 2\theta}$$

$$\cos 4\theta = \frac{\sin 8\theta}{2\sin 4\theta}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cos 2^{n-1} \,.\, \theta = \frac{\sin 2^n \,.\, \theta}{2\sin^{n-1} \,.\, \theta};$$

$$\therefore\ \cos\theta \,.\, \cos 2\theta \,.\, \cos 4\theta \ldots\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \,.\, \sin\theta}.$$

Now for θ substitute 8θ, and let $n = 6$.

$$\text{Then } \sin 8\theta \,.\, \cos 8\theta \,.\, \cos 16\theta \ldots\ldots\ldots \cos 256\theta = \frac{\sin 512\theta \,.\, \sin 8\theta}{64 \,.\, \sin 8\theta}$$

$$= \frac{\sin 512\theta}{64}$$

$$= \frac{-1}{64} \text{ by hypothesis,}$$

$$\therefore\ \sin\ 512\theta = -1 = \sin 270 \quad (12)$$

$$\text{and } 512\theta = 270.$$

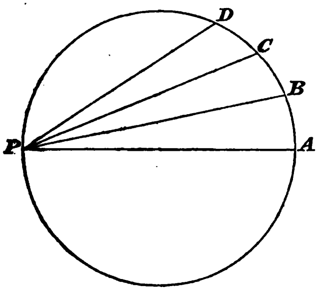
$$\text{and } \theta = \frac{270}{512} = \frac{135}{256}$$

$$= \overset{\text{i}}{31} \,.\, \overset{\text{ii}}{38} \,.\, \overset{\text{iii}}{26} \,.\, \overset{\text{iv}}{15}.$$

202. $\dfrac{\text{chd}\,2\theta}{\text{chd}\,\theta} - \dfrac{\text{chd}\,\theta}{\text{chd}\,2\theta} = \dfrac{\text{chd}\,3\theta}{\text{chd}\,2\theta}$. Prove this by Vieta's property of chords.

$$\text{Let arc}\,AB = \text{arc}\,BC = \text{arc}\,CD = \theta;$$

$$\therefore\ AC = 2\theta \text{ and } AD = 3\theta.$$



Draw the diameter AP and the chords PB, PC, PD.

$$\text{Then } PB = \text{chd}\,(\pi - \theta),$$

$$PC = \text{chd}\,(\pi - 2\theta),$$

$$PD = \text{chd}\,(\pi - 3\theta).$$

$$\text{Let } PB = x + x^{-1}.$$

Then by Vieta's property,

$$PC = x + x^{-2},$$

$$\text{and } PD = x + x^{-3}.$$

$$\text{Now } \sqrt{(4 - \text{chd}^2\,\theta)} = \text{chd}\,(\pi - \theta) \quad (9)$$

$$= x + x^{-1};$$

$$\therefore\ 4 - \text{chd}^2\,\theta = x^2 + 2 + x^{-2};$$

$$\therefore\ \text{chd}^2\,\theta = 4 - 2 - (x^2 + x^{-2}) = 2 - (x^2 + x^{-2}) = -(x^2 - 2 + x^{-2}) = -(x - x^{-1})^2;$$

$$\therefore\ \text{chd}\,\theta = \sqrt{-1}\,.\,(x - x^{-1}).$$

$$\text{Similarly chd}\,2\theta = \sqrt{-1}\,.\,(x^2 - x^{-2}),$$

$$\text{chd}\,3\theta = \sqrt{-1}\,(x^3 - x^{-3});$$

$$\therefore \frac{\text{chd}\,2\theta}{\text{chd}\,\theta} - \frac{\text{chd}\,\theta}{\text{chd}\,2\theta} = \frac{x^2 - x^{-2}}{x - x^{-1}} - \frac{(x - x^{-1})}{x^2 - x^{-2}}$$

$$= \frac{(x^2 - x^{-2})^2 - (x - x^{-1})^2}{(x - x^{-1})\,(x^2 - x^{-2})}$$

$$= \frac{(x^4 - 2 + x^{-4}) - (x - x^{-1})^2}{(x - x^{-1})\,(x^2 - x^{-2})}$$

$$= \frac{(x^3 + x - x^{-1} - x^{-3}) - (x - x^{-1})}{x^2 - x^{-2}}$$

$$= \frac{x^3 - x^{-3}}{x^2 - x^{-2}}$$

$$= \frac{\text{chd}\,3\theta}{\text{chd}\,2\theta}.$$

203. $$\frac{\theta}{1.2.3} - \frac{2\theta^3}{1.2.3.4.5} + \frac{3\theta^5}{1.2.3.4.5.6.7} - \frac{4\theta^7}{1.2.3.4.5.6.7.8.9} + \ldots = \frac{\sin\theta - \theta\,.\cos\theta}{2\theta^2},$$

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5.6} + \ldots\ldots \quad (63)$$

$$= 1 - \frac{3\theta^2}{1.2.3} + \frac{5\theta^4}{1.2.3.4.5} - \frac{7\theta^6}{1.2.3.4.5.6.7} + \ldots\ldots$$

$$\therefore\ \theta\,.\cos\theta = \theta - \frac{3\theta^3}{1.2.3} + \frac{5\theta^5}{1.2.3.4.5} - \frac{7\theta^7}{1.2.3.4.5.6.7} + \ldots\ldots$$

$$\text{But } \sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \frac{\theta^7}{1.2.3.4.5.6.7} + \ldots.(62);$$

$$\therefore\ \sin\theta - \theta\,.\cos\theta = \frac{2\theta^3}{1.2.3} - \frac{4\theta^5}{1.2.3.4.5} + \frac{6\theta^7}{1.2.3.4.5.6.7} - \ldots\ldots$$

$$\therefore\ \frac{\sin\theta - \theta\,.\cos\theta}{2\theta^2} = \frac{\theta}{1.2.3} - \frac{2\theta^3}{1.2.3.4.5} + \frac{3\theta^5}{1.2.3.4.5.6.7} - \ldots\ldots$$

204. If θ be *very small,* $3\log\theta = \log.\tan\theta + 2\log.\text{chd}\,\theta.$

$$\tan\theta = \theta + \frac{\theta^3}{1.3} + \frac{2\theta^5}{1.3.5} + \ldots\ldots\ldots \quad (64)$$

$=\theta+\dfrac{\theta^3}{1.3}$ (neglecting the higher powers of θ as indefinitely small)

$=\theta+\dfrac{2\theta^3}{6}$,

and $\sin\theta=\theta-\dfrac{\theta^3}{1.2.3}+\dfrac{\theta^5}{1.2.3.4.5}-\ldots\ldots\ldots$ (62)

$=\theta-\dfrac{\theta^3}{6}$ (neglecting the higher powers of θ as before);

$$\therefore\ \tan\theta-\sin\theta=\frac{3\theta^3}{6}=\frac{\theta^3}{2};$$

$$\therefore\ \theta^3=2(\tan\theta-\sin\theta)$$

$$=2\tan\theta.(1-\cos\theta)$$

$$=4\tan\theta.\sin^2\frac{\theta}{2}$$

$$=\tan\theta.\text{chd}^2\theta;$$

$$\therefore\ 3\log\theta=\log.\tan\theta+2\log.\text{chd}\,\theta.$$

205. If θ be *very small*, then $2\log\theta=\log.\tan\theta+\log.\sec\theta$.

$\cos\theta=1-\dfrac{\theta^2}{1.2}+\dfrac{\theta^4}{1.2.3.4}-\ldots\ldots\ldots$ (63)

$=1-\dfrac{\theta^2}{1.2}$,

and $\sec\theta=1+\dfrac{\theta^2}{1.2}+\dfrac{5\theta^4}{1.2.3.4}+\ldots\ldots\ldots$ (65)

$=1+\dfrac{\theta^2}{1.2}$;

$$\therefore\ \theta^2=\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta}=\tan\theta.\sec\theta;$$

$$\therefore\ 2\log\theta=\log.\tan\theta+\log.\sec\theta.$$

Cor. From this and the preceding theorem it immediately appears that $5\log\theta=2(\log.\text{chd}\,\theta+\log.\tan\theta)+\log.\sec\theta$.

206. $\frac{1}{2}\cdot\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+\frac{1}{4}\cdot\left(\cos^4\frac{\theta}{2}-\sin^4\frac{\theta}{2}\right)$

$$+\frac{1}{6}\cdot\left(\cos^6\frac{\theta}{2}-\sin^6\frac{\theta}{2}\right)+\dots\dots\dots=\cos\theta+\frac{1}{3}\cdot\cos^3\theta+\frac{1}{5}\cdot\cos^5\theta+\dots\dots\dots$$

$$\frac{1}{2}\cdot\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+\frac{1}{4}\cdot\left(\cos^4\frac{\theta}{2}-\sin^4\frac{\theta}{2}\right)+\frac{1}{6}\cdot\left(\cos^6\frac{\theta}{2}-\sin^6\frac{\theta}{2}\right)+\dots\dots\dots$$

$$=\frac{1}{2}\cdot\left(-\sin^2\frac{\theta}{2}-\frac{1}{2}\cdot\sin^4\frac{\theta}{2}-\dots\dots\dots\right)-\frac{1}{2}\cdot\left(-\cos^2\frac{\theta}{2}-\frac{1}{2}\cdot\cos^4\frac{\theta}{2}-\dots\dots\dots\right)$$

$$=\frac{1}{2}\cdot\log_e\left(1-\sin^2\frac{\theta}{2}\right)-\frac{1}{2}\cdot\log_e\left(1-\cos^2\frac{\theta}{2}\right)\quad(75)$$

$$=\frac{1}{2}\cdot\log_e.\cos^2\frac{\theta}{2}-\frac{1}{2}\cdot\log_e.\sin^2\frac{\theta}{2}$$

$$=\frac{1}{2}\cdot\log_e\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\quad(32)\text{ and }(72)$$

$$=\frac{1}{2}\cdot\log_e(1+\cos\theta)-\frac{1}{2}\cdot\log_e(1-\cos\theta)\quad(72)$$

$$=\frac{1}{2}\cdot\left(\cos\theta-\frac{1}{2}\cdot\cos^2\theta+\frac{1}{3}\cdot\cos^3\theta-\dots\dots\dots\right)$$

$$-\frac{1}{2}\left(-\cos\theta-\frac{1}{2}\cos^2\theta-\frac{1}{3}\cdot\cos^3\theta-\dots\dots\dots\right)\quad(75)$$

$$=\cos\theta+\frac{1}{3}\cdot\cos^3\theta+\frac{1}{5}\cdot\cos^5\theta+\dots\dots\dots$$

207. $\operatorname{vers}3\theta.\sin\theta-\frac{1}{2}.\operatorname{vers}6\theta.\sin2\theta+\frac{1}{3}.\operatorname{vers}9\theta.\sin3\theta-\dots\dots=0.$

$$\operatorname{vers}3\theta.\sin\theta=(1-\cos3\theta).\sin\theta=\sin\theta-\sin\theta.\cos3\theta$$

$$=\sin\theta-\frac{1}{2}.(2\cos3\theta.\sin\theta),$$

$$\frac{1}{2}.\operatorname{vers}6\theta.\sin2\theta=(1-\cos6\theta).\frac{1}{2}.\sin2\theta=\frac{1}{2}.\sin2\theta-\frac{1}{2}.\sin2\theta.\cos6\theta$$

$$=\frac{1}{2}.\sin2\theta-\frac{1}{2}.\left(\frac{1}{2}.2\cos6\theta.\sin2\theta\right),$$

$$\frac{1}{3}\cdot\text{vers}\,9\theta\,.\,\sin 3\theta=(1-\cos 9\theta)\,.\,\frac{1}{3}\,.\,\sin 3\theta=\frac{1}{3}\,.\,\sin 3\theta-\frac{1}{3}\,.\,\sin 3\theta\,.\cos 9\theta$$

$$=\frac{1}{3}\,.\sin 3\theta-\frac{1}{2}\,.\left(\frac{1}{3}\,.\,2\cos 9\theta\,.\,\sin 3\theta\right),$$

……

$$\therefore\ \text{vers}\,3\theta\,.\,\sin\theta-\frac{1}{2}\,.\,\text{vers}\,6\theta\,.\,\sin 2\theta+\frac{1}{3}\,.\,\text{vers}\,9\theta\,.\,\sin 3\theta-\ldots\ldots\ldots$$

$$=\left(\sin\theta-\frac{1}{2}\,.\sin 2\theta+\frac{1}{3}\,.\sin 3\theta-\ldots\ldots\ldots\right)$$

$$-\frac{1}{2}\,.\left\{(2\cos 3\theta\,.\,\sin\theta)-\frac{1}{2}\,.\,(2\cos 6\theta\,.\sin 2\theta)+\frac{1}{3}\,.\,(2\cos 9\theta\,.\,\sin 3\theta)-\ldots\ldots\right\}$$

$$=\frac{\theta}{2}-\frac{1}{2}\,.\,\{(\sin 4\theta-\sin 2\theta)-\frac{1}{2}\,.\,(\sin 8\theta-\sin 4\theta)$$

$$+\frac{1}{3}\,.\,(\sin 12\theta-\sin 6\theta)-\ldots\ldots\ldots\}\quad\text{(61) and (40)}$$

$$=\frac{\theta}{2}-\frac{1}{2}\,.\,(\sin 4\theta-\frac{1}{2}\,.\sin 8\theta+\frac{1}{3}\,.\sin 12\theta-\ldots\ldots\ldots)$$

$$+\frac{1}{2}\,.\,(\sin 2\theta-\frac{1}{2}\,.\sin 4\theta+\frac{1}{3}\,.\sin 6\theta-\ldots\ldots\ldots)$$

$$=\frac{1}{2}\,.\,\theta-\frac{1}{2}\,.\left(\frac{4\theta}{2}\right)+\frac{1}{2}\,.\left(\frac{2\theta}{2}\right)\quad(61)$$

$$=\frac{\theta}{2}-\theta+\frac{\theta}{2}$$

$$=\theta-\theta$$

$$=0.$$